EXPONENTIAL FUNCTIONS : GROWTH AND DECAY

In the next few class sessions, we will investigate a class of processes that are described by mathematically by exponential functions. There are a number of processes that occur in nature that involve either exponential growth or decay, including the rate at which atmospheric pressure decreases with height above the surface of the earth, the rate at which radioactive nuclei decay, the rate at which money grows at a constant interest rate, or the rate at which the value of money decreases at a constant inflation rate.

As you can see, understanding exponential growth and decay is invaluable in investigating many of the phenomena we observe in nature.

An exponential relationship will occur whenever a continuous rate of change of some quantity (say, the amount of money you have) is proportional to the amount of the quantity that currently exists. To see how exponential process work, let' s consider a simplified model for bank interest. We will use specific number to allow us to do calculations, but the basic ideas will hold for any situation.

For instance, suppose you are told a bank will pay 1 % interest each day on your money (and good luck getting this interest rate these days). This means that if you have $1000 in the bank today, at the end of the day they will add 1 % to your total so that at the end of day 1 you will have a sum of :

Total funds at end of day 1 = funds at end of day 0 + 1 % × funds at end of day zero

or you will have at the end of day 1 :

funds at end of day 1 = $1000 + 0.01×$1000 = $1010.

At the end of day 2, your funds will increase by an additional 1 %. Since you have slightly more money now than the day before, the 1 % interest will generate a slightly greater dividend (since you are taking 1 % of a slightly larger base amount). So at the end of day 2, you will have :

funds at end of day 2 = $1010 + 0.01×$1010 = $1020 .10

Continuing this line of reasoning we we have :

funds at end of day 3 = $1020 .10 + 0.01× $1020 .10 = $1030 .30
funds at end of day 4 = $1030 .30 + 0.01× $1030 .30 = $1040 .60

You can see that each day the amount of interest added to your account increases slightly; for these first few days that amount of daily increase in interest seems miniscule (10 cents the first day, 20 cents the second day, 30 cents the third day), but if we continued this process for several months (and did not withdraw any money), we would have $2000 in the bank after 70 days; in fact, if money could increase by 1 % per day (a huge interest rate in fact), your money would double in value every 70 days. ***(About how much money would you have in a year?) Estimate how long it would take to to accumulate $10, 000 (without withdrawing funds)?***

In each case, the rate of change of money (that is, how much money is added to your account each day) is proportional to the amount of money already in your account. Examples like this show us the power of exponential increase (or decrease).

We can analyze our example in a slightly different way. We know that at the end of the first day the total amount of money in your account is described as :

funds at the end of the first day = \$1000 + 0.01×\$1000

We can write this as \$1000 (1 + 0.01) = \$1000 (1.01)

At the end of the second day, we will have am amount of money equal to :

$$\$1000\,(1.01) + 0.01\,(\$1000)\,(1.01) = \$1000\,(1.01)\,(1 + 0.01) =$$
$$\$1000\,(1.01)\,(1.01) = \$1000\,(1.01)^2.$$

Use this logic to write an equation for the amount of money at the end of the third and fourth days, can you generalize your results to the amount of money that will exist at the end of n days?

Now suppose instead of giving you interest every day, the bank gave you interest every second, or even every millisecond. This very unrealistic banking example would approach a continuous rate of change. If your money increased by 1 % at the end of each second, we could describe the total amount of money you have at any time by using the exponential number :

$$M(t) = \$1000\, e^{0.01\, t}$$

where M (t) is the amount of money you have at any time, \$1000 is the amount of money you had when you started (when t = 0), t is the amount of time (measured in secodns) that has elapsed, and e is the exponential number. The exponential number, e, has a numerical value (accurate to 4 decimal places) of 2.718. If your calculator has a button labeled e or e^x, this is the button you use to do calculations with the exponential number e.

The exponential number follows all the familiar rules for multiplication :

$$e \times e = e^2 \; (2.718 \times 2.718 = 7.3875)$$
$$e^2 \times e = e^3 = 2.718^3 = 20.0855$$
$$e^{-1} = 1/e = 1/2.718 = 0.3679$$
$$e^{-2} = 1/e^2 = 1/2.718^2 = 0.1354$$

The M & M Activity

Tonight in class we will do an activity that will illustrate the mathematics underlying exponential processes. We will be investigating a form of exponential decay, using M & M candies.

I will give each person a bag of M & Ms so there will be four bags at each table.

First, count up how many M & Ms you have in total at your table and record this number. Then, shaking up the M & Ms vigorously, release them on the table (try to keep them from falling onto the floor) and count the number of M & Ms that land face up (that is, with the "M" facing up); count the number that landed face up and discard the remainder . (Before you do your count, what is your prediction for the number that will land face up, what is your reasoning for this prediction?) (some write - ups say you can eat them; I would not recommend this for two reasons : 1) each bag contains 200 calories (and after all it is close to dinner time) and 2) we do biology labs in this room so I cannot promise that the table is the most immaculate eating surface imaginable. If you really are intent on eating your M & Ms, first place down some paper towels and release the candy onto them. This is likely a good idea even if you don' t want to eat the discards since it will keep the M & Ms from bouncing around too much.)

Repeat this process; shake the M & Ms vigorously and release them on the table, count the number facing up and discard the rest. Record this data; continue this process until there are no surviving M & Ms.

For next week : Submit your data table (showing how many M & Ms survived after each round); your initial number should be considered round 0.

Construct a graph of number of surviving M & Ms vs. round number (starting with 0), and draw the curve that best approximates the data points.

Answer the following questions :

1) Why do you think it is an important step to vigorously shake the M & Ms?

2) Suppose we had an odd batch of M & Ms that were weighted such that there was an 80 % probability of each candy landing face up? If you started with the same number of candies you had in this expt, construct a data table showing your predictions for how many candies would survive each round. Estimate how many rounds it would take before all the candies were discarded.

3) Repeat this set of calculations in the event that each candy has a 25 % probability of landing face up.

Keep a copy of your data; we will use them next week.