DERIVING RELATIONSHIPS AMONG DISTANCE, SPEED AND ACCELERATION

For the last several weeks, we have been doing investigations in class designed to demonstrate the properties of objects in motion. You have conducted several types of experiments, measured motion data, and used that data to determine the value of such parameters as final instantaneous speed and acceleration.

Tonight we will consider some of the more mathematical aspects of this work and learn how to use and manipulate equations to derive these relationships.

Basic Concepts

Let' s start by reviewing some basic concepts. The average speed of an object is the ratio of the total distance it travels to the total time of travel; in equation form this is :

$$\text{average speed} = \frac{\text{total distance}}{\text{time traveled}} = \frac{d}{t} \tag{1}$$

You have all calculated the average speed for a cart rolling down a ramp in a number of different experiments. We have also been introduced to the concept of *instantaneous* speed, the speed of an object at a specific point in time. (You can think of this as the speed shown on a car' s speedometer at the moment you observe it.) So let' s say an object is traveling at 20 m/s when it enters a street and is traveling at 30 m/s when it exits the street. Even though we don' t know either the total distance or total time traveled, we can find the average velocity by using the simple definition of average :

$$\text{average of two quantities} = \frac{\text{value of quantity 1} + \text{value of quantity 2}}{2}$$

Or in this case,

$$\text{average speed} = \frac{\text{speed 1} + \text{speed 2}}{2} = \frac{20\,\text{m/s} + 30\,\text{m/s}}{2} = 25\,\text{m/s} \tag{2}$$

Let' s rewrite this using some more standard scientific notation; we would likely call the speed of the car entering the street as its initial speed, and designate that as v_i where the subscript "i" stands for initial, and we would read this as "v sub i". (Sometimes the subscript "o" is used also: v_o and is read as "v sub zero" or "v naught".)

The speed of the car on exiting the street is the final speed and designated as v_f, or "v sub f". Using this notation, we write eq. (2) as:

$$v_{av} = \frac{v_i + v_f}{2} \text{ where } v_{av} = \text{average speed} \tag{3}$$

We also studied the concept of acceleration, and learned that for straight line motion, acceleration is defined as the rate of change of speed, or :

$$\text{acceleration} = \frac{\text{change in speed}}{\text{time}} = \frac{v_f - v_i}{t} \tag{4}$$

Remember that in science, we always compute the change in a quantity by subtracting the initial value of the quantity from its final value.

Combining equations

We have now expressed three very basic equations of motion; the defining equations for average speed and acceleration. Suppose we have a situation similar to the experiments you have conducted over the last few weeks where we are careful to start the object from rest. We are interested in finding out how far the object will travel in a certain elapsed time, t if it is accelerating with a rate expressed as a. In this case (where our object starts from rest), our initial velocity is zero and our equations become :

$$v_{av} = \frac{v_i + v_f}{2} = \frac{v_f}{2} \qquad \text{if } v_i = 0 \tag{5}$$

$$\text{acceleration} = a = \frac{v_f}{t} \quad \text{if } v_i = 0 \tag{6}$$

Now, since eq. (1) tells us that:

$$v_{av} = \frac{d}{t} \tag{7}$$

and eq. (2) tells us when the initial speed = 0 that :

$$v_{av} = \frac{v_f}{2} \tag{8}$$

Eqs. (7) and (8) are just different ways of writing the average speed; therefore the terms on the right sides of eqs. (7) and (8) are equal to each other, and we are completely correct in writing :

$$\frac{d}{t} = \frac{v_f}{2} \tag{9}$$

In eq. (9), if we multiply each side by a factor of t, we obtain :

$$t \times \frac{d}{t} = \frac{v_f}{2} \times t$$

or :

$$d = \frac{v_f\ t}{2} \tag{10}$$

Now, we have expressed the distance traveled as a function of final instantaneous speed and time. But look at eq. (6); this equation links the value of the acceleration to the final speed, so if we multiply each side of eq. (6) by time (t), we get :

$$t \times a = \frac{v_f}{t} \times t$$

or :

$$v_f = at \tag{11}$$

Finally, notice that eq. (11) gives us an expression for v_f and also that v_f appears in eq. (10). This means that we can use the expression for v_f in eq. (11) and substitute that expression in place of v_f where it appears in eq. (10). This gives us:

$$\mathbf{d = \frac{(a\,t)\,t}{2} = \frac{1}{2}\,a\,t^2} \tag{12}$$

This final equation shows us the power of combining equations; by starting with very basic definitions of speed and acceleration, we can derive the expression telling us how far an accelerating object will travel if it starts from rest, and has an acceleration whose value is given by a.

This is one of the most common and important *quadratic* relationships in nature. It tells you that if an accelerating object travels 1 unit of distance in 1 unit of time, the object will travel **4 units of distance in two units of time.**

Question : How much farther will an accelerating object travel in 4 time units compared to 1 time unit?

Question : If an accelerating object travels one meter in one second, how long would it take to travel 4 meters? To travel 25 meters? To travel 1/2 meter?

To do in class : Using data from your experiment last week, determine the acceleration of your cart down the ramp.

To do in class : In today' s Winter Olympics Women' s Downhill event, Julia Mancuso attained a speed of 27.41 m/s at the first intermediate position which she reached 41.14 s after starting from rest, and had attained a speed of 30.00 m/s at the second intermediae position after a total elapsed time of 1 : 44.75 (1

minute and 44.75 s). What was her acceleration during the first 41.14 s of travel? What was her acceleration between the first and second intermediate positions?

To do in class : Consider now an object that does not start from rest, this means we cannot set $v_i=0$. Use the appropriate forms of the equations for average speed and acceleration and derive the equation that relates distance traveled to initial speed, acceleration and time of travel.