GRAPHING FOURIER SERIES

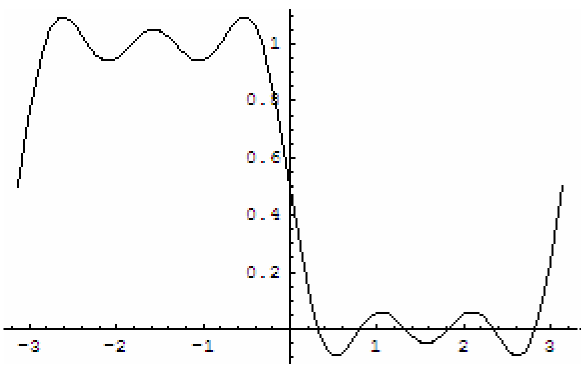
As an addendum to homework set #8 (computations of Fourier series), this write up will show graphically how Fourier series converge to given functions.

Problem 1 on this homework set asks for the Fourier expansion of the function defined as 1 on (-π, 0) and as 0 on (0, π).

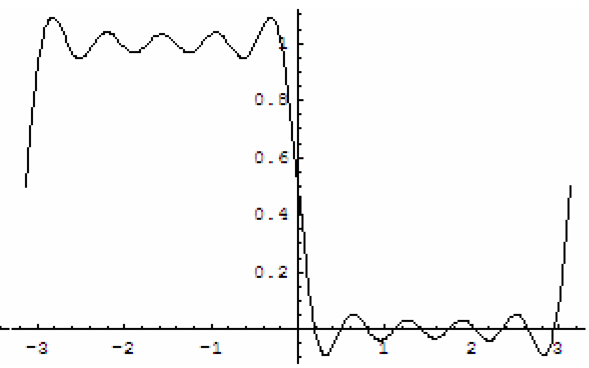
The Fourier series for this function is:

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1} \frac{\sin(2n-1)x}{2n-1}$$

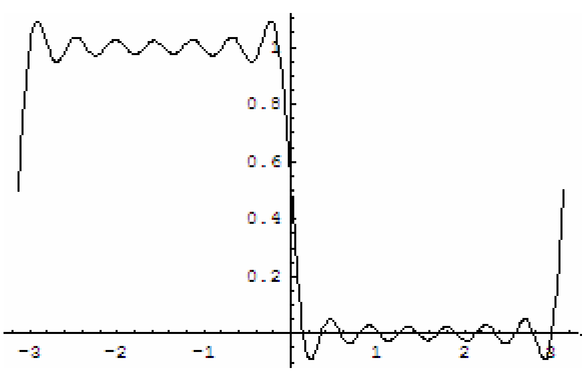
The following four graphs show the function evaluated with different numbers of terms in the expansion:



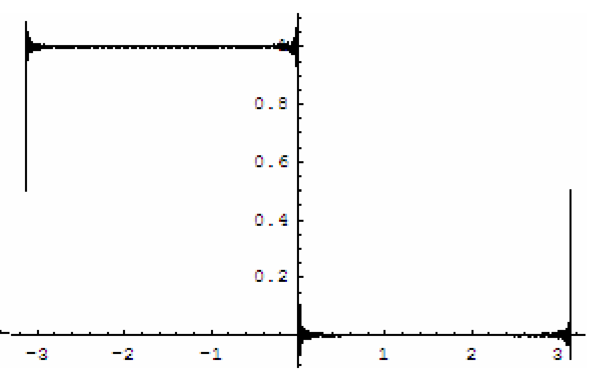
The function with three terms of the expansion



The function with five terms of the expansion



with seven terms of the expansion



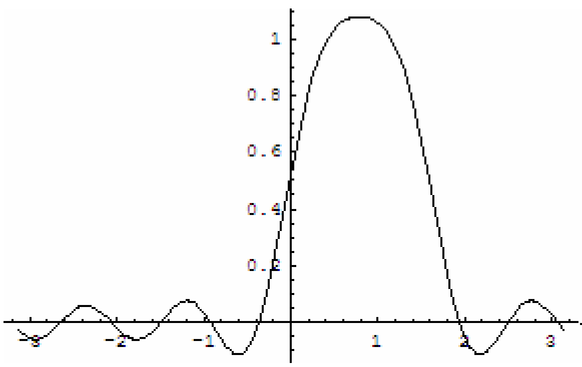
and for grins, with 100 terms

The second homework problem seeks the Fourier expansion of a function with value equal to 1 on (0, π/2) and equal to 0 elsewhere on (-π, π). We can write this function analytically as:

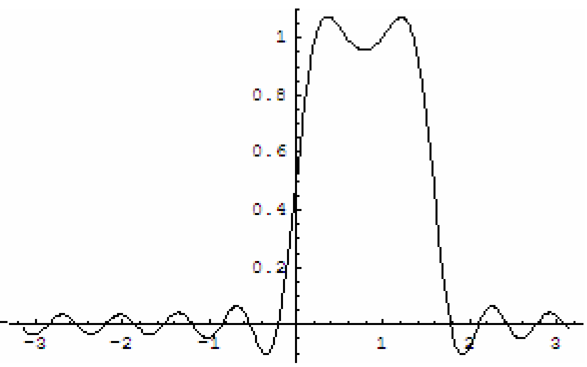
$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1} \frac{(-1)^{n+1}\cos(2n-1)x}{(2n-1)} + \frac{1}{\pi}\sum_{n=1} \frac{\sin(nx)}{n} + \frac{1}{\pi}\sum_{n=1} \frac{\sin(4n-2)x}{4n-2} - \frac{1}{\pi}\sum_{n=1} \frac{\sin(4n)x}{4n}$$

where the last two terms take of the fact that the coefficient of sin(nx) is 2 for n=2, 6, 10, …, and also that all terms with n=4k vanish.

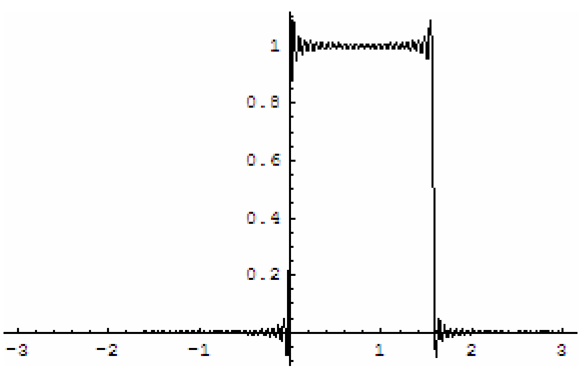
The graphs of this function are:



including terms through cos(5x) and sin(5x)



including terms through cos(9x) and sin (9x)



and just because we can, here is the graph of the function for coefficients through n=100.

We can see in these two cases how the Fourier series converge to the defined function.

In class today, we looked at cases where we had functions that did not meet the Dirichlet condition's; in other words, we looked at functions that were defined on some interval other than (-L, L). We know that if a function is defined on (-L, L) (and meets the other conditions of Dirichlet) then we can express that function as a Fourier series, and that the Fourier series we compute will converge to the function on the defined interval.

In class, we considered the function defined as:

$$f(x) = \begin{cases} 1, & 0 < x < 1/2 \\ 0, & 1/2 < x \end{cases} \qquad (1)$$

While this is a perfectly well defined function, we cannot as yet expand it in a Fourier series since it does not meet the Dirichlet condition of being 2L periodic. However, we can make this series 2L periodic by extending it along the negative *x* axis. We can make this extension almost anything we want, but we can make our lives easier if we choose our extensions to make *f(x)*either an even or odd function. (This simplifies our lives because we can make use of the symmetry arguments we covered in class today).

Let's proceed as we did in class, first by making the extension of this function along the negative *x* axis in such a way as to produce an odd function.

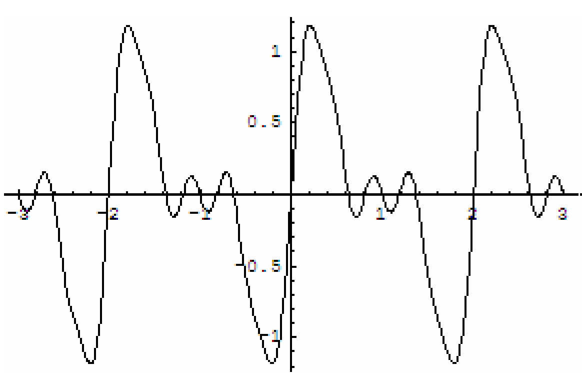
If f is now odd, we know all the a_n coefficients vanish, and we are left only with b_n coefficients, which we calculated in class to be:

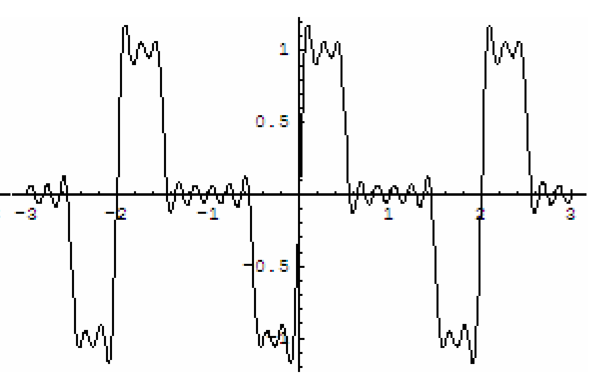
$$b_n = \frac{2}{n\pi}(1 - \cos(n\pi/2)) = \begin{cases} 0 \quad n = 4, 8, 12 \\ \frac{4}{n\pi} \, n = 2,\ 6,\ 10 \\ \frac{2}{n\pi} \ n = odd \end{cases}$$

The Fourier sine series then becomes:

$$f(x) = \frac{2}{\pi}[\sin(\pi x) + \frac{2\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \frac{2\sin(6\pi x)}{6} + ...]$$

Graphically, the function looks like:

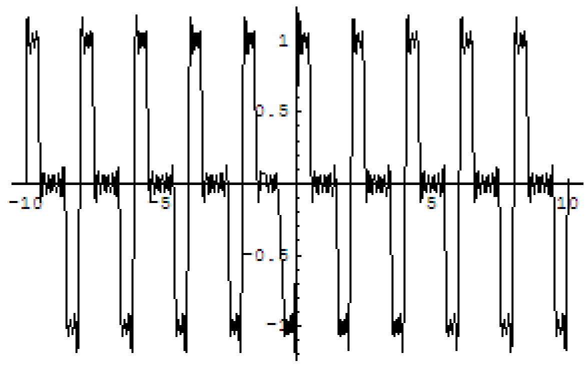




including terms through n=5

including terms through n = 10

Notice how the Fourier expansion reproduces the required behavior of the function on the original interval (0,1). Note also how this Fourier expansion produces an odd function that is periodic on (-1,1), and repeats with a period of 2. We can see the periodic behavior very well if I extend the range of the plot to (-10,10):



through terms n=10

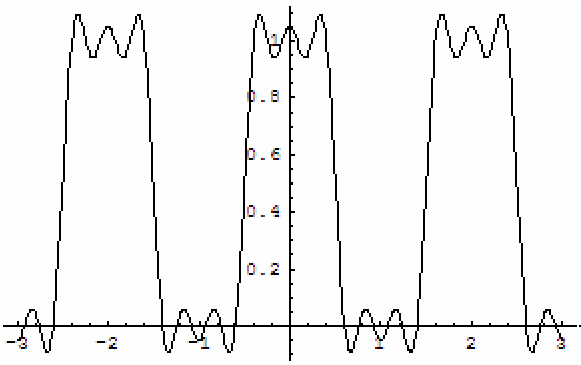
Now, let's take the same function and extend it into the negative half-plane by assuming it is an even function. As we did in class, we found that a_0=1, and the a_n terms are given by:

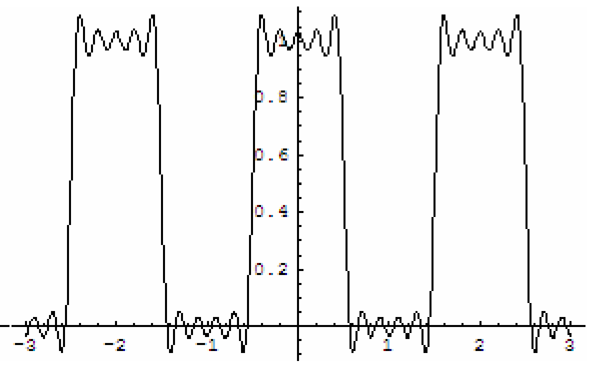
$$a_n = \frac{2}{n\pi}\sin(n\pi/2) = \begin{cases} 0 \ n \ even \\ \frac{2}{n\pi} n = 1, 5, 9 \\ \frac{-2}{n\pi} \ n = 3, 7, 11 \end{cases}$$

This gives us a Fourier cosine expansion:

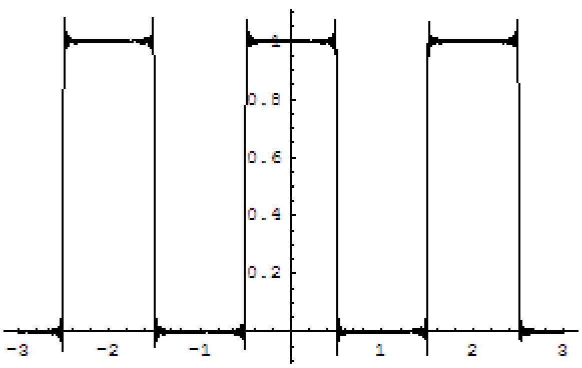
$$f(x) = \frac{1}{2} + \frac{2}{\pi}[\cos(\pi x) - \frac{\cos(3\pi x)}{3} + \frac{\cos(5\pi x)}{5} - ...]$$

the graph of this function is:





through terms n=5 through terms n = 10



and for grins, through n=100
(notice that there is still a “spike” at the discontinuities; this is called the Gibbs’ Phenomenon and is mentioned briefly in the text)

Notice that the graph of the cosine series is different from that of the sine series, but that the two graphs both reproduce the function as defined in (1) above.

And extending the graph over several cycles, we obtain (again, n=100 terms)