USING SYMMETRY TO EVALUATE INTEGRALS

Evaluation of Fourier series requires evaluating integrals of trig functions to determine the coefficients of the sin and cos terms of the Fourier series. Many of these integrals appear daunting or at least irritating at first glance. However, we can reduce our effort and time spent doing integrations if we make use of symmetry arguments in evaluating these integrals.

Let's look at some integrals that are commonly encountered in Fourier analysis. For instance, if we are asked to find the Fourier series for f(x) = x on the interval {-π, π}, we would have to evaluate the integrals

$$\int_{-\pi}^{\pi} x\cos(nx)dx \quad and \quad \int_{-\pi}^{\pi} x\sin(nx)dx$$

to determine the coefficients a_n and b_n in the Fourier expansion.

Our first instinct in evaluating these integrals is likely to integrate by parts, but before we dive headlong into integration, let's take a step back and look at the nature of the integrand in each case.

In the first integral, the integrand is a product of x and cos(nx); we know that x is an odd function on the interval of integration, and that cos(nx) is an even function. (Recall that the definition of an even function is that f(x) = f(-x); a condition met by cos(nx). Also, we can remember that the Taylor series expansion of cos(x) involves only even powers of x.)

The product of an even function and an odd function is an odd function, and the integral of an odd function integrated over an interval symmetric about the origin is zero. Try integrating a few odd functions like x^n over an interval symmetric about the origin, i.e, an interval {-L, L}.

Now it is clear to us that the integral of x cos(nx) over any interval symmetric about the origin is zero, and that all a_n terms in the Fourier expansion of f(x)=x vanish. This is not too surprising; we would expect the expansion of an odd function like f(x)=x to include only odd terms.

The second integral above involves the product of two odd functions; the product of two odd functions is an even function. {Consider two odd functions, g(x) and h(x). If they are both odd they both have the property that f(x) = -f(-x), so g(-x)h(-x) = [-g(x)][-h(x)]=g(x)h(x), and the product of these functions is even.}

Knowing the integrand is even, we can make the sometimes useful simplification that:

$$\int_{-\pi}^{\pi} x\sin(nx)dx = 2\int_{0}^{\pi} x\sin(nx)dx$$

Sec. 7.9 of Boas makes use of this simplification. For other cases where you can make use of symmetry arguments to help in evaluating Fourier series, look at problem 23 on page 371 of Boas. If we represent the plucked string as a periodic function on {0, L}, is that function odd, even (or neither)? Would you expect the Fourier series for this function to be composed of only sin or cos terms (or mixed terms)?

Let's look at some other integrals you might encounter in your future courses. In statistical mechanics and thermodynamics, you will study the dynamics of gases, and in particular, you will have to evaluate integrals involving Gaussian terms, i.e., terms of the form *exp(-x^2)*. I showed in a previous classnote (from Feb. 17) how to evaluate the integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx \qquad (1)$$

The value of this integral over the entire x axis is simply $\sqrt{\pi}$.

The study of kinetic theory will require evaluation of various moments of the Gaussian function, i.e., integrals of the form:

$$\int_{-\infty}^{\infty} x^n \exp(-x^2) dx \quad (2)$$

Let's look at this integral for n=1. Before we begin integrating by parts, we recognize that the integrand is a product of an odd function (*x*) and an even function (exp(-x^2)). Thus, integral (2) represents an odd function integrated over a symmetric interval. Using our symmetry arguments from before, we know we can set this integral equal to zero.

How would we handle integrals of this form for even values of n? For n = 2 we have the integral:

$$\int_{-\infty}^{\infty} x^2 \exp(-x^2) dx \qquad (3)$$

It is clear that we should use integration by parts, but what do we set equal to *u* and *dv* in (4) below? (Here, I am using the notation for integration by parts such that:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du \quad (4)$$

We can solve this integral rather easily by making the less than profound observation that $x^2 = x \cdot x$, so that we can write (3) as:

$$\int_{-\infty}^{\infty} x(x \exp(-x^2))dx \quad (5)$$

written in this way, it is clear that we should set $u = x$ and $dv = xexp(-x^2)$, leaving us with:

$$\int_{-\infty}^{\infty} x^2 \exp(-x^2)dx = x(-\frac{1}{2}\exp(-x^2))|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \exp(-x^2)dx \qquad (6)$$

The right hand side of (6) is very easy to evaluate. Since $exp(-x^2)$ goes to zero much faster than x as $x \to \infty$, the first term on the right is zero. We recognize the second term as ½ the Gaussian integral (whose value we know to be $\pi^{1/2}$), so the value of the integral in (3) is simply $\frac{\sqrt{\pi}}{2}$.

We can extend this reasoning to consider all integrals of the form $\int_{-\infty}^{\infty} x^n \exp(-x^2)dx$.

For odd values of n, we can make use of the same symmetry arguments as used before to show the integral vanishes over the indicated limits.

For even powers of n, we can integrate by parts, this time making the somewhat more profound observation that $x^n = x \cdot x^{n-1}$. With this observation in hand, we can write the integral as:

$$\int_{-\infty}^{\infty} x^n \exp(-x^2)dx = \int_{-\infty}^{\infty} x^{n-1}[x \exp(-x^2)]dx = x^{n-1}(-\frac{1}{2}\exp(-x^2))|_{-\infty}^{\infty} + \frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2} \exp(-x^2)dx \qquad (7)$$

And of course your first instinct is 'oh great', since after all this work we just recover another integral involving the exp($-x^2$) term times a power of x. But if we look at this carefully, we can figure out how to evaluate these integrals without having to do any further explicit integration.

If we look on the right hand side of (7), we know the first term vanishes when evaluated at $\pm\infty$, leaving us only with the unsolved integral. However, we already know about integrals of this form; *n* must be even for the integral to be non-zero, and if that is the case, then *n-2* is also even. So we can integrate this integral by parts until we get to an integral of the form $\int_{-\infty}^{\infty} \exp(-x^2)dx$ (multiplied by some coefficient, of course).

Whenever we integrate $\int_{-\infty}^{\infty} x^n \exp(-x^2)dx$ by parts, the only nonvanishing part of the integral is the $\frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2} \exp(-x^2)dx$ term. Our goal is to iteratively integrate by parts until we reach the case of $n = 2$, so that we are left only with the Gaussian integral as shown in (1). So, let's say we are asked to find the value of $\int_{-\infty}^{\infty} x^6 \exp(-x^2)dx$, we proceed as:

$$\int_{-\infty}^{\infty} x^6 \exp(-x^2)dx = \frac{5}{2}\int x^4 \exp(-x^2)dx =$$

$$\frac{5}{2}\cdot\frac{3}{2}\int x^2 \exp(-x^2)dx = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\int \exp(-x^2)dx = \frac{15}{8}\sqrt{\pi}$$

Note that I have omitted the limits in the integrals; all limits should be understood to be $\pm\infty$.

How would we evaluate $\int_{-\infty}^{\infty} x^8 \exp(-x^2)dx$? First, use integration by parts to obtain:

$$\int_{-\infty}^{\infty} x^8 \exp(-x^2)dx = \frac{n-1}{2}\int_{-\infty}^{\infty} x^{n-2} \exp(-x^2)dx = \frac{7}{2}\int_{-\infty}^{\infty} x^6 \exp(-x^2)dx = \frac{7}{2}\frac{15}{8}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}$$

See if you can figure out the general solution for integrals of the form $\int_{-\infty}^{\infty} x^n \exp(-x^2)dx$.