DERIVING RELATIONSHIPS

Some Worked Examples

As we have seen in class (and on homework assignments), deriving new relationships from previously known equations is an important enterprise in physics. In this classnote, I use the basic equations of projectile motion and show how we can derive new equations.

In these examples, let' s consider a projectile fired on level ground with an initial velocity of v_o making an angle θ with the ground. As in all cases, we neglect the effects of air friction.

Suppose we can observe the maximum height reached by the projectile and its range (horizontal distance traveled). These data can be determined by observers some distance from the launch. Can we use these observables and our knowledge of the value of g to determine the initial velocity and the launch angle? Our task is to construct equations for v_0 and θ in terms of R (range), H (maximum height) and g.

We begin by recalling two equations we can easily determine from the equations of motion. The range of a projectile is given by :

$$R = \frac{2\, v_0^2 \sin\theta \cos\theta}{g} \tag{1}$$

and the maximum height above the launch point is :

$$H = \frac{v_0^2 \sin^2\theta}{2\, g} \tag{2}$$

If we want to express initial velocity in terms of R, H and g only, we need to find an equation that does not involve θ in any way. In other words, we have to find ways to eliminate sin θ and cos θ from eq. (1) To do this, let' s begin by rewriting eq. (2) as :

$$\sin^2\theta = \frac{2\, g\, H}{v_o^2} \tag{3}$$

Taking the square root of both sides of eq. (3) yields :

$$\sin\theta = \frac{\sqrt{2\, g\, H}}{v_o} \tag{4}$$

and we could subsitute this into eq. (1) in place of sin θ. But, eq. (1) still has a cos θ term. Since we want to write v_o in terms of only R, H and g, we have to think of a way to eliminate the cos θ term. We can do this by recalling the trig identity :

$$\sin^2\theta + \cos^2\theta = 1 \;\Rightarrow\; \cos^2\theta = 1 - \sin^2\theta \;\Rightarrow\; \cos\theta = \sqrt{1 - \sin^2\theta} \tag{5}$$

Now, we take the result for sin θ in eq. (3) and substitute into eq. (5) :

$$\cos\theta = \sqrt{1 - \frac{2\,g\,H}{v_0^2}} = \sqrt{\frac{v_0^2 - 2\,g\,H}{v_o^2}} = \frac{\sqrt{v_o^2 - 2\,g\,H}}{v_o} \tag{6}$$

Make sure you can follow all the algebraic steps in eq. (6) above.

Now, we have expressions for sin θ and cos θ that we can substitute back into eq. (1). Using the expresssion for sin θ from eq. (4) and the expression for cos θ fro eq. (6), our initial equation (1) becomes :

$$R = \frac{2\,v_0^2}{g}\,\frac{\sqrt{2\,g\,H}}{v_o}\,\frac{\sqrt{v_o^2 - 2\,g\,H}}{v_o} = \frac{2\sqrt{2\,g\,H}\,\sqrt{v_o^2 - 2\,g\,H}}{g} \tag{7}$$

Look carefully at eq. (7) above : notice that the equation includes only the variables R, H, g and the initial velocity. We are almost done; all we have to do now is rewrite eq. (7) so that we express v_o in terms of R, H and g. In the words of basic algebra, our job is to isolate v_o on one side of the equation.

Start by multiplying each side of eq. (7) by g and squaring both sides :

$$g^2\,R^2 = 4\,(2\,g\,H)\left(v_o^2 - 2\,gH\right)$$

Divide by 8 g H :

$$\frac{g^2\,R^2}{8\,g\,H} = v_o^2 - 2\,g\,H$$

Add 2 g H to both sides of the equation and add fractions :

$$v_o^2 = \frac{g^2\,R^2}{8\,g\,H} + 2\,g\,H = \frac{g^2\,R^2 + 16\,g^2\,H^2}{8\,g\,H} = \frac{g\,R^2 + 16\,g\,H^2}{8\,H} = \frac{g\left(R^2 + 16\,H^2\right)}{8\,H}$$

Finally, take the square root and we can write v as a function only of the observables R, H and the known value of g :

$$v_o = \sqrt{\frac{g\left(R^2 + 16\,H^2\right)}{8\,H}} \tag{8}$$

Now, suppose we would like to find the launch angle θ in terms of R, H and g. We take the expression for v_o in eq. (8) and substitute into eq. (4) :

$$\sin\theta = \frac{\sqrt{2\,g\,H}}{\left(\sqrt{\frac{g\left(R^2+16\,H^2\right)}{8\,H}}\right)} = \frac{\sqrt{8\,H}\,\sqrt{2\,g\,H}}{\sqrt{g\left(R^2 + 16\,H^2\right)}} = \frac{\sqrt{16\,g\,H^2}}{\sqrt{g\left(R^2 + 16\,H^2\right)}} = \frac{4\,H}{\sqrt{R^2 + 16\,H^2}}$$

In the final step above, notice that there is a common factor of g in both square roots which cancel out. We could leave our answer in this form, or take the inverse sin to obtain :

$$\theta = \sin^{-1}\left(\frac{4\,H}{\sqrt{R^2 + 16\,H^2}}\right) \tag{9}$$

Are these the exact steps that you should follow to do the homework assignments? No, but these are good examples to show you how you can use equations you already know to derive new equations.

I recommend that you study these derivations carefully, making absolutely sure you understand each step before going on to the next. You will know you understand this if you can put away your notes (and not look at this write up) and derive the equations for yourself, using only the basic equations of motion and the necessary trig identities.