PHYS 111
HOMEWORK #8

Due : 3 November 2016

1. As we saw in class, an oscillating mass connected to a spring will have a period of oscillation that is determined by the mass of the object and the stiffness of the spring. In other words, the period of oscillation can be written as :

$$P = c\, k^a\, m^b$$

where c is a constant, k is the spring constant (units of N/m) and m is the mass (units of kg). Use the techniques of dimensional analysis to determine the coefficients a and b, and thus the equation for the period of an oscillating mass on a spring.

Solution : Period has units of time (seconds); the spring constant k has units of N/m or writing in terms of the fundamental SI units (which are kg, m and s) kg m s^{-2}/m = kg s^{-2}. Mass has units of kg. Combining all these units and using the laws of exponents, we get:

$$s = \left(kg\, s^{-2}\right)^a (kg)^b = kg^a\, s^{-2a}\, kg^b = kg^{a+b}\, s^{-2a}$$

The exponent of seconds on the left hand side is 1, so the exponent of seconds on the right hand side is also 1. This means that:

$$-2a = 1 \Rightarrow a = \frac{-1}{2}$$

The left hand side only involves units of seconds, so the exponent of kg on the left must be zero. This means that

$$kg^{a+b} = kg^0 \Rightarrow a + b = 0 \Rightarrow b = -a = \frac{1}{2}$$

Thus, a = -1/2, b = +1/2, and our equation for the period of an oscillating spring is :

$$P = c\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{k}}$$

(This analysis will not yield the value of the constant; I provide the value for compleness)

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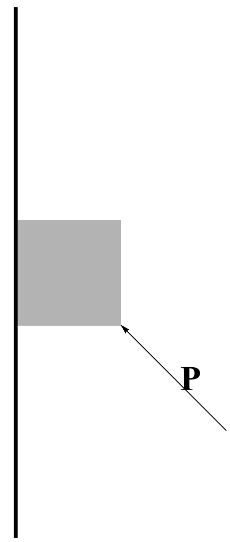
2. A block of mass m is held against a vertical wall by a force P (see diagram below). The coefficients of static and kinetic friction between the mass and wall are μ_s and μ_k respectively. P makes an angle θ with respect to the

vertical. What value of P will cause the mass to move up the wall at a constant speed?



Solution : This problem shows us an example of a normal force that does not depend on the weight of the object. The forces acting on the mass are its weight (down), friction (down), the normal force of the wall (to the right), and the pushing force P (with a component to the left and a component up).

If the mass moves at a constant speed, there is no acceleration and all forces sum to zero in both the x and y directions. Applying the second law to the block we have:

$$\Sigma F_x = N - P\sin\theta = 0 \Rightarrow N = P\sin\theta$$

There is a normal force of the wall pushing against the box, but it does not related to the weight of the box since in this case, the weight is perpendicular to the normal force. If the box is moving up at a constant speed, we can write (setting up as the positive direction) :

$$\Sigma F_y = P\cos\theta - W - f_k = 0$$

The frictional force is described by $f_k = \mu_k N = \mu_k P \sin\theta$. Using this expression for f_k we can rewrite the sum of forces in the y direction as:

$$\Sigma F_y = P\cos\theta - W - \mu_k P\sin\theta = 0$$

Regrouping :

$$P(\cos\theta - \mu_k \sin\theta) = W \Rightarrow P = \frac{W}{\cos\theta - \mu_k \sin\theta}$$

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3. A block slides down an inclined plane of angle θ at constant speed. It is then projected up the same plane with an initial speed of v_o. How far up the plane will it travel before coming to rest? (Your answer should be in terms of v_o, g and θ (but not μ)). Will it slide down again?

Solution : Let' s solve this problem in reverse; our ultimate goal is to find the maximum distance the block will slide up a plane. We will use the kinematic equation :

$$v_f^2 = v_o^2 + 2\,a\,s \tag{1}$$

We know v_o; we know that $v_f = 0$, and we want to find the path length s along the ramp. To do this, we will need an expression for the acceleration of the block up the ramp.

We know that as the block slides up the ramp, it will experience a force due to the component of gravity down the plane, and also a force due to friction. If we write Newton's second law for the block for forces parallel to the plane, we get:

$$\Sigma F_{||} = -\,m\,g\sin\theta - \mu_k\,m\,g\cos\theta = m\,a \Rightarrow a = -g\,(\sin\theta + \mu_k\cos\theta) \tag{2}$$

We have set the positive direction to be up the plane; both the component of gravity and friction act down the plane. This gives us an expression for a, but it involves the unknown μ_k. We can obtain an expression for μ_k by considering the blocks' s motion down the plane. We are told that the block slides down the plane at a constant speed. If in this case we set down the plane as the positive direction, we write:

$$\Sigma F_{||} = m\,g\sin\theta - \mu_k\,m\,g\cos\theta = 0 \Rightarrow m\,g\sin\theta = \mu_k\,m\,g\cos\theta \Rightarrow \mu_k = \tan\theta.$$

Using this expression for μ_k in equation (2) gives us:

$$a = -\,g\,(\sin\theta + \tan\theta\cos\theta)$$

But since $\tan\theta = \sin\theta/\cos\theta$, the product of $\tan\theta\cos\theta = \sin\theta$, and our acceleration is simply:

$$a = -\,2\,g\sin\theta$$

Using this value of a in equation (1):

$$v_f^2 = v_o^2 + 2\,(-2\,g\sin\theta)\,s \Rightarrow 0 = v_o^2 - 4\,g\sin\theta\,s$$

or :

$$s = \frac{v_o^2}{4\,g\sin\theta}$$

Once the block reaches this maximum distance up the plane, will it slide back down? No, it won't. Let's see why.

The statement of the problem tells us that the block slid down the plane at a constant speed. This means that the component of gravity down the plane ($m\,g\sin\theta$) is equal to the force of kinetic friction (f_k). For the stationary block to slide back down the plane, we will need the component of gravity to be greater than the force of static friction (f_s). However, since the coefficient of static friction is greater than the coefficient of kinetic friction ($\mu_s > \mu_k$), the component of gravity down the plane cannot exceed the force of static friction (since, as stated before, the component of gravity down the plane is equal to the kinetic friction force). Mathematically we can express this as:

$$f_s > f_k = m\,g\sin\theta \Rightarrow f_s > m\,g\sin\theta$$

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4. Problem 64, p. 151 text.

Solution : In part a), the masses move at constant speed. This tells us two things : 1) we use the coefficient of kinetic friction, and 2) the sum of forces in each direction is zero since there is no acceleration. We begin as always with Newton' s second Law :

The forces acting on m_2 are the tension in the string and its weight, thus, the force balance for m_2 is simple:

$$T - m_2 g = 0 \Rightarrow T = m_2 g$$

Since the string is masselss and the pulley is frictionless, the tension will be the same throughout the string. The forces along the plane acting on m_1 are the tension (up the plane), and gravity and friction (down the plane). This means that:

$$T - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = 0$$

Recalling that T = m_2 g, we can write:

$$m_2 g = m_1 g (\sin\theta + \mu_k \cos\theta)$$

or:

$$m_2 = m_1 (\sin\theta + \mu_k \cos)\theta$$

And this is the value of the hanging mass that will allow the mass on the ramp to slide up the ramp at constant speed once the system is set into motion.

In part b), we are asked to consider the case of the block sliding down the plane at constant speed. The difference with part a) is that now, the force of friction acts up the plane. This yields the equation (for forces on m_1):

$$T - m_1 g \sin\theta + \mu_k m g \cos\theta = 0$$

Since T = m_2 g (the force equation for m_2 is unchanged), we get:

$$m_2 g = m_1 g (\sin\theta - \mu_k \cos\theta) \Rightarrow m_2 = m_1 (\sin\theta - \mu_k \cos\theta)$$

In part c), we are asked to find the range of masses of m_2 that would cause the system to remain at rest. We want to make sure m_2 is not so massive that it accelerates the system downward, nor so light that m_1 pulls the system down the ramp. We would write Newton's second laws for each mass; they would look exactly as they do in parts a) and b) except in this case we are using the coefficient of static friction (since there is no motion). Thus the range of masses for m_2 that will keep the system at rest are:

$$m_1 (\sin\theta - \mu_s \cos\theta) < m_2 < m_1 (\sin\theta + \mu_s \cos\theta)$$

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5. Problem 70, p. 151 text.

Solution : In order for the block to remain stationary and not slide down, the force of friction between the block and the cart must be equal to the weight of the block. The force of friction is :

$$f_s = \mu_s N$$

and the force of friction must equal the weight of the mass m. The subtle part of this problem in in computing the normal force. The statemet of the problem tells us that the cart M rolls without friction. This means that if we apply a force to M, the cart (and the small mass m) will accelerate in the direction of the force.

Our goal is to determine an expression for the normal force (and thus the frictional force). To do so, we have to analyze the forces acting on the mass m and also the mass M.

Let's consider the horizontal forces acting on m. There is a force F acting to the left (which we will call the positive direction), and a normal force N between m and M that acts on m to the right. We can write this as:

$$\Sigma F_m = F - N = m a \Rightarrow N = F - m a \tag{3}$$

where a is the acceleration of m. Now, if m is held to M, we know they both accelerate at the same rate. The only force acting on M is the normal force between m and M (F is not in contact with M so does not enter into its second law equation):

$$\Sigma F_M = N = M a \Rightarrow a = \frac{N}{M} \tag{4}$$

Using this expression for a in equation (3) above:

$$N = F - m a = F - m\left(\frac{N}{M}\right) \Rightarrow N\left(1 + \frac{m}{M}\right) = F$$

or

$$N = \frac{F}{1 + \frac{m}{M}}$$

Since the frictional force between m and M must equal the weight of m, we can write:

$$f_s = \mu_s N = \frac{\mu_s F}{1 + \frac{m}{M}} = m g \Rightarrow F = \frac{\left(1 + \frac{m}{M}\right) m g}{\mu_S}$$

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6. Read the bridging problem section on the bottom of p. 172. Draw a free body diagram for the mass if the cylinder is rotating. Answer the two questions posed in the write up (questions a) and b)). 20 points for this problem

Solutions : In the first part, the block is stationary against the inside of the cylinder wall. Let' s choose our coordinates to be the vertical direction and the radial direction. The forces acting on the block are gravity (down), friction (up), and the normal force due to the wall of the cylinder (acting toward the center of the rotation). The block does not slide vertically, so the sum of vertical forces is zero. However, since the cylinder is rotating, there is an acceleration toward the rotation axis. The relevant force equations become :

$$\text{(Sum of forces vertically)}: \; f_s - m\,g = 0 \Rightarrow f_s = m\,g \tag{5}$$

$$\text{(Sum of forces radially)}: \; N = \frac{m\,v^2}{r} \tag{6}$$

$$\text{(Friction)}: \; f_s = \mu_s\,N \tag{7}$$

We can use the expression for N in eq. (6) to rewrite eq. (7):

$$f_s = \mu_s \frac{m\,v^2}{r}$$

and use this in eq. (5) to obtain:

$$\frac{\mu_s\,m\,v^2}{r} = m\,g \Rightarrow v^2 = \frac{g\,r}{\mu_s} \tag{8}$$

This is the minimum speed of rotation that will keep the mass from sliding down. The book asks us to find the period of this rotation. One period is the time to make one rotation; as we remember from kinematics:

$$\text{speed} = \frac{\text{distance}}{\text{time}} \Rightarrow v = \frac{2\,\pi\,r}{P} \Rightarrow v^2 = \frac{4\,\pi^2\,r^2}{P^2}$$

Using this expression in eq. (6):

$$\frac{4\,\pi^2\,r^2}{P^2} = \frac{g\,r}{\mu_s} \Rightarrow P^2 = \frac{4\,\pi^2\,r^2\,\mu_s}{g\,r} \Rightarrow P = 2\,\pi\sqrt{\frac{\mu_s\,r}{g}}$$

where I use P to represent the period of rotation. (Verify that the units are correct.)

In part b), the rotation is slowed so that the block accelerates down the cylinder wall. The relevant equations change slightly:

$$\Sigma F_{rad} = N = \frac{m\,v^2}{r}$$

$$\Sigma F_y = m\,g - f_k = m\,a$$

(I am setting down as the positive direction, and remember we use kinetic friction since there is motion)

$$f_k = \mu_k\,N$$

Combining these equations yields:

$$m\,a = m\,g - \mu_k \frac{m\,v^2}{r} \Rightarrow a = g - \frac{\mu_k\,v^2}{r}$$

If our period is now designated as P_f, we have that

$$v^2 = \frac{4\pi^2 r^2}{P_f^2}$$

and :

$$a = g - \frac{4\pi^2 r \mu_k}{P_f^2}$$

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7. Problem 4, p. 175 text.

Solution : Let' s solve the general problem (i.e., using only symbols rather than numbers). This way we will only have to do one calculation at the end. We start by considering the forces on the car. In the vertical direction, the normal force of the unbanked road equals the weight of the car, so we can write :

$$N = m g$$

The frictional force between the tires and the road exert a radial force on the car, keeping the car from "sliding" off the road. This means that friction allows the car to turn, and not simply move in a straight line tangent to the curve. Friction is the only force acting radially, and it is the force that produces the centripetal acceleration of the car. We can express this as:

$$\Sigma F_{rad} = f_s = \frac{m v^2}{r}$$

Finally, we know that $f_s = \mu_s N = \mu_s m g = \frac{m v^2}{r} \Rightarrow v = \sqrt{\mu_s r g}$

This relationship tells us the maximum safe speed for a curve of radius r and coefficient of friction μ. If we double the coefficient of friction, the safe speed will increase by a factor of $\sqrt{2}$ or the new safe speed is 1.41 times the original speed.. If we decrease the coefficient of friction by a factor of 2, the safe speed decreases by a factor of $\sqrt{2}$, or the new safe speed is 0.71 times as much as the original speed.

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8. Problem 5, p. 175 text.

Solution : The forces acting on the seats are the tension in the rod, and the weight of the person acting down. The tension in the rod has both horizontal and vertical components, and we know the sum of forces must produce a centripetal acceleration directed toward the rotation axis. For this problem, our most convenient set of coordinate axes are a vertical axis, and a radial axis (directed toward the center of the circle).

Since the acceleration is only radial and not vertical, we know the sum of vertical forces is equal. Calling T the tension in the rod, we have :

$$\Sigma F_{vertical} = T \cos\theta = m g \tag{9}$$

where θ is the angle of the rod from the vertical. (In the problem, $\theta = 30^o$, but let's solve this analytically before we substitute numbers.)

The only force acting radially is the horizontal component of force, and this must produce the centripetal acceleration. This gives us:

$$\Sigma F_{\text{radial}} = T \sin\theta = \frac{m v^2}{r} \tag{10}$$

where r is the radius of the orbit around the rotation axis. We are asked to find the period of revolution. To do so, we need to find the speed of the passenger. We can do this by dividing equation (8) by equation (7):

$$\frac{T \sin\theta}{T \cos\theta} = \frac{\frac{m v^2}{r}}{m g} \Rightarrow \tan\theta = \frac{v^2}{r g}$$

or :

$$v = \sqrt{r g \tan\theta}$$

Notice that the angle of the arm does not involve the mass of the person, so that the answer to part c) is no. To find the period of revolution, we recall that period (the time to make one orbit) is equal to the circumference of the orbit divided by the speed of motion, or:

$$P = \frac{2 \pi r}{v} = \frac{2 \pi r}{\sqrt{r g \tan\theta}} = 2 \pi \sqrt{\frac{r}{g \tan\theta}}$$

and make sure that you verify that the dimensions match. Now, we can find the period of revolution, but we must take care to compute r. We are told that the horizontal arm at the top is 3 m across, and the passenger seat is at the end of a 5 m rod that makes an angle of 30^o with the vertical. The distance of the passenger from the central axis is then:

$$r = 3 \text{ m} + 5 \sin 30 \text{ m} = 5.5 \text{ m}$$

Then, using the data in the problem, we have :

$$P = 2 \pi \sqrt{\frac{5.5 \text{ m}}{\left(9.8 \text{ m s}^{-2}\right) \tan 30}} = 6.19 \text{ s}$$

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9. Problem 8, p. 175 text.

Solution : Refer to Fig. 6.8 on p. 158 of the text. Follow the derivation in the text (and in classnotes). For this problem, we use a standard x - y set of Cartesian axes (and not the vertical/radial set of other problems invoving rotation). The reason we use the x - y axes is that we need to find components of the normal force. It is also important to remember that in this problem, all of the force needed to produce the centripetal acceleration derives from the horizontal component of the normal force (and none from friction between the wheels and the road). Therefore, the only forces acting on the car are its weight and the normal force (which has both horizontal and vertical components). The vertical component of the normal force must equal the weight of the car, and the horizontal component of the normal force produces the centripetal acceleration. There is no acceleration in the vertical direction. In Newton's second law terms, these results give us the equations:

$$\Sigma F_y = N\cos\theta - mg = 0 \Rightarrow N\cos\theta = m\,g$$

$$\Sigma F_x = N\sin\theta = \frac{m\,v^2}{r}$$

Dividing equations :

$$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{m\,v^2}{r}}{m\,g} \Rightarrow \tan\theta = \frac{v^2}{r\,g}$$

We are asked to find the proper angle to bank a curve of 900 ft radius so that cars can safely navigate the curve at 55mi/hr. Converting to SI units:

$$900\text{ ft} = 900\text{ ft}\,(1\text{ meter}/3.28\text{ ft}) = 274\text{ m}$$

$$55\text{ mi}/\text{hr} = 55\text{ mi}/\text{hr}\,(5280\text{ ft}/\text{mi})\,(1\text{ m}/3.28\text{ ft})\,(1\text{ hr}/3600\text{ s}) = 24.6\text{ m}/\text{s}$$

Therefore,

$$\tan\theta = \frac{(24.6\text{ m}/\text{s})^2}{\left(274\text{ m}\cdot 9.8\text{ m s}^{-2}\right)} = 0.22 \Rightarrow \theta = 12.7^{\text{o}}$$

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10. Problem 10, p. 175 text.

Solution : Again with the bowling ball. In this case, we are asked to find the acceleration of the ball and the tension in the rope at the moment the ball is at its lowest point. This is actually a much simpler problem than most of the ones we have done, since it involves forces only in one direction (the vertical direction). At the moment the ball reaches its nadir (word for lowest point), the forces acting on it are the tension (up) and weight (down). These two forces produce the centripetal acceleration of the ball. Casting this into the form of Newton' s second law, we get :

$$T - m\,g = \frac{mv^2}{r}$$

The acceleration acts along the direction of the rope (straight toward the center of revolution) and has magnitude m v^2/r = $(4.2\ m/s)^2$/3.8m = 4.64m s^{-2}. The mass of a 71.2N object is 7.3kg, so we have:

$$T = m\,g + \frac{m\,v^2}{r} = 71.2\text{ N} + 7.3\text{ kg}\cdot 4.64\text{ ms}^{-2} = 105\text{ N}$$