Introduction to Legendre Polynomials

We began recently our study of the Legendre differential equation. We will discover that the solutions to these differential equations are a set of functions known as the Legendre polynomials. These polynomials are defined on [-1, 1], and have a number of useful and interesting mathematical properties.

Basic Properties

We can call Legendre polynomials in Mathematica using :

```
LegendreP[n, x]
```

where n represents the nth polynomial, and x is the variable. To write out the first ten Legendre polynomials :

```
In[136]:= Do[Print["The n = ", n,
    "  Legendre polynomial is:  ", LegendreP[n, x]], {n, 0, 9}]
```

The n = 0 Legendre polynomial is: 1

The n = 1 Legendre polynomial is: x

The n = 2 Legendre polynomial is: $\frac{1}{2}\left(-1+3x^2\right)$

The n = 3 Legendre polynomial is: $\frac{1}{2}\left(-3x+5x^3\right)$

The n = 4 Legendre polynomial is: $\frac{1}{8}\left(3-30x^2+35x^4\right)$

The n = 5 Legendre polynomial is: $\frac{1}{8}\left(15x-70x^3+63x^5\right)$

The n = 6 Legendre polynomial is: $\frac{1}{16}\left(-5+105x^2-315x^4+231x^6\right)$

The n = 7 Legendre polynomial is: $\frac{1}{16}\left(-35x+315x^3-693x^5+429x^7\right)$

The n = 8 Legendre polynomial is:

$$\frac{1}{128}\left(35-1260x^2+6930x^4-12\,012x^6+6435x^8\right)$$

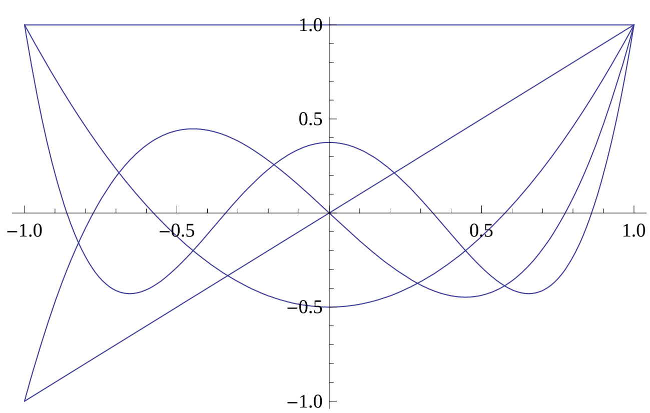
The n = 9 Legendre polynomial is:

$$\frac{1}{128}\left(315x-4620x^3+18\,018x^5-25\,740x^7+12\,155x^9\right)$$

We can plot the first several polynomials :

In[135]:=
```
Plot[LegendreP[n, x] /. n → {0, 1, 2, 3, 4}, {x, -1, 1}]
```

Out[135]=



How many roots does the n^{th} Legendre polynomial have.

Notice that the even order Legendre polynomials are even, and the odd orders are odd functions.

Legendre polynomials are orthogonal on (-1, 1). Integrate any two Legendre polynomials of different orders and obtain :

In[137]:=
```
Integrate[LegendreP[2, x] LegendreP[3, x], {x, -1, 1}]
```

Out[137]= 0

But if the polynomials are of the same order, we get :

In[142]:=
```
Integrate[LegendreP[2, x] LegendreP[2, x], {x, -1, 1}]
```

Out[142]= $\frac{2}{5}$

In[143]:=
```
Integrate[LegendreP[3, x] LegendreP[3, x], {x, -1, 1}]
```

Out[143]= $\frac{2}{7}$

Legendre polynomials are defined to be orthonormal, meaning the integral of a product of Legendre polynomials is either zero or one. In other words, there is an orthonormal constant, N, such that :

$$N \int_{-1}^{1} P_n(x)\, P_n(x)\, dx = 1$$

Find an expression for N.

Solutions to differential equations :

We are learning that Legendre polynomials are one of the solutions to the Legendre differential equation :

$$\left(1 - x^2\right) y'' - 2\, x\, y' + m\,(m+1)\, y = 0$$

If we solve this differential equation using DSolve in Mathematica, we obtain :

In[152]:=
```
DSolve[(1 - x^2) y''[x] - 2 x y'[x] + m (m + 1) y[x] == 0, y[x], x]
```

Out[152]=
```
{{y[x] → C[1] LegendreP[m, x] + C[2] LegendreQ[m, x]}}
```

As expected, we get two solutions to a second order differential equation. The two solutions are the Legendre polynomials of the first kind (LegendreP[m, x]) and Legendre polynomials of the second kind (LegendreQ[m, x]). We will spend our time studying the former, since those solutions converge everywhere on [-1, 1]. The polynomials of the second kind may not converge (as described in Boas, 12.2), as we can see by calculating the value of the third order Legendre polynomial of the second kind at x = 1 and - 1 :

In[156]:=
```
LegendreQ[3, x] /. x → {-1, 1}
```

Out[156]= {∞, ∞}

If we specify a value for the constant in the Legendre differential equation, we obtain for our solutions :

In[157]:=
```
m = 4;
DSolve[(1 - x^2) y''[x] - 2 x y'[x] + m (m + 1) y[x] == 0, y[x], x]
```

Out[158]=
$$\left\{\left\{y[x] \to \frac{1}{8}\left(3 - 30x^2 + 35x^4\right) C[1] + C[2] \left(\frac{55x}{24} - \frac{35x^3}{8} + \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)\left(-\frac{1}{2}\,\mathrm{Log}[1 - x] + \frac{1}{2}\,\mathrm{Log}[1 + x]\right)\right)\right\}\right\}$$

Notice that the first solution is just the explicit form of LegendreP function; the second solution is the corresponding polynomial of the second kind.

Since we are interested in functions that converge on [-1,1], we are interested only in the Legendre polynomials of the first kind. For instance, when we solve Laplace's equaiton on a sphere, we want solutions that will be valid at the north and south poles (whose polar coordinates are cos θ =±1), therefore, the physically meaningful solutions to Laplace's equation on a sphere are the polynomials of the first kind.