# POLAR COORDINATES AND CELESTIAL MECHANICS

In class, we showed that the acceleration vector in plane polar (r, $\phi$) coordinates can be written as :

$$\mathbf{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{1}$$

where r is the distance from the origin and $\phi$ is the azimuthal angle.

If we apply eq. (1) to the orbits of planets in the solar system, we know that the force causing the acceleration is the gravitational force, so that we can write :

$$\mathbf{F} = m\,\mathbf{a} = -\left(G\,m\,M/r^2\right)\hat{r} \tag{2}$$

where G is the Newtonian gravitational constant and M is the mass of the central object. It is important to note that F depends only on distance from the central object (in our analysis, this is the sun), and does not depend at all on the azimuthal angle. If we compare the expressions for **a** in eqs. (1) and (2), we realize that the $\phi$ component is zero in eq. (2), so that we can write for motion in a gravitational field :

$$\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}} \Rightarrow r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \tag{3}$$

We showed in class that

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \tag{4}$$

And we can verify this statement via :

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = \frac{1}{r}\left(2\,r\,\dot{r}\,\dot{\phi} + r^2\ddot{\phi}\right) = 2\,\dot{r}\,\dot{\phi} + r\ddot{\phi} \tag{5}$$

Now, equation (4) tells us that

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 \tag{6}$$

which implies that the quantity $r^2 \dot{\phi}$ is a constant. You might recognize $r^2 \dot{\phi}$ as angular momentum per unit mass; the analysis we have just done is one way to prove the conservation of angular momentum for planetary orbits. In celestial mechanics, we usually call this constant h, so that we will use for the rest of our analysis:

$$h = r^2 \dot{\phi} \tag{7}$$

The comparison of eqs. (1) and (2) also allow us to equate the radial components of the acceleration vector, telling us that :

$$m\left(\ddot{r} - r\dot{\phi}^2\right) = -G\, m\, M \big/ r^2 \tag{8}$$

We can utilize eq. (7) to write $\dot{\phi} = h/r^2$. Square this expression and substitute into eq. (8) to obtain:

$$\ddot{r} - h^2 \big/ r^3 = -G\, M \big/ r^2 \tag{9}$$

This differential equation describes the distance of the planet from the sun as a function of time. What we really want to have to pinpoint the location of the planet in space is a relationship between the distance of the planet and its azimuthal angle in its orbit. In other words, we want to know the (r,$\phi$) components of an orbiting object. Also, eq. (9) above is fairly complicated to solve; and we would like an equation whose solutions are a bit more obvious.

***Making the proper substitution :***

We will accomplish both of our goals of the last paragraph by making the substitution :

$$u = \frac{1}{r} \tag{10}$$

and also by recalling that the conservation of angular momentum provides :

$$h = r^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{h}{r^2} = h\, u^2 \tag{11}$$

So, we want to convert differential equation (9) into an equation in terms of u and $\phi$. To do this, we will need to transform the $d^2 r \big/ dt^2$ term into an experssion involving u and $\phi$. Our first step in this process using the chain rule to show that:

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{h}{r^2} \frac{dr}{d\phi} \tag{12}$$

the last step occurring by virtue of eq. (11). Now, we use the chain rule to analyze the dr/d$\phi$ term :

$$\frac{dr}{d\phi} = \frac{dr}{du}\,\frac{du}{d\phi} = \frac{-1}{u^2}\,\frac{du}{d\phi} \tag{13}$$

Substituting this into the dr/d$\phi$ term in eq. (12) gives us for dr/dt :

$$\frac{dr}{dt} = \frac{h}{r^2}\,\frac{dr}{d\phi} = hu^2\left(\frac{-1}{u^2}\,\frac{du}{d\phi}\right) = -h\,\frac{du}{d\phi} \tag{14}$$

Now, we use eq. (14) :

$$\ddot{r} \equiv \frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(-h\,\frac{du}{d\phi}\right) = \frac{d}{d\phi}\left(-h\,\frac{du}{d\phi}\right)\frac{d\phi}{dt} \tag{15}$$

where the last step in eq. (15) is the result of the chain rule. Using eq. (11) in eq. (15) and differentiating, we have :

$$\ddot{r} = \frac{d}{d\phi}\left(-h\,\frac{du}{d\phi}\right)\frac{d\phi}{dt} = -h\,\frac{d^2 u}{d\phi^2}\cdot h\,u^2 = -h^2\,u^2\,\frac{d^2 u}{d\phi^2} \tag{16}$$

We have now expressions for $\ddot{r}$ and r that we can substitute into eq. (9),
which gives us the equation of motion :

$$-h^2\,u^2\,\frac{d^2 u}{d\phi^2} - h^2\,u^3 = -G\,M\,u^2 \tag{17}$$

Dividing through by - h $u^2$ yields:

$$\mathbf{\frac{d^2 u}{d\phi^2} + u = \frac{G\,M}{h^2}} \tag{18}$$

Equation (18) will be our basic equation of celestial mechanics.

Looking at eq. (18), you should notice that it is a familiar differential equation; the left hand side has the same form as the harmonic oscillator equation. Very simple techniques of solving differential equations allow us to write :

$$u = A\cos\phi + B\sin\phi + G\,M\big/h^2 \tag{19}$$

or since r = 1/u,

$$r = \frac{1}{A\cos\phi + B\sin\phi + G\,M\big/h^2} \tag{20}$$

***A Little Bit of Trig***

We will now do a bit of trigonometric massage on eq. (20) to put it into an even easier form to use, and the form that you are familiar with if you have studied planetary motion in any advanced courses.

First, we choose a constant C such that :

$$C = \sqrt{A^2 + B^2}, \tag{21}$$

and we choose an angle $\alpha$ such that

$$\tan\alpha = \frac{B}{A} \tag{22}$$

This definition of $\alpha$ implies that

$$\sin\alpha = \frac{B}{\sqrt{A^2 + B^2}} \text{ and } \cos\alpha = \frac{A}{\sqrt{A^2 + B^2}} \tag{23}$$

Armed with these tools of trigonometric destruction, we take the first two terms on the right of eq. (19) and write them as :

$$A\cos\phi + B\sin\phi = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\cos\phi + \frac{B}{\sqrt{A^2 + B^2}}\sin\phi\right) \tag{24}$$

The step above just multiplied and divided each term in the equation by $\sqrt{A^2 + B^2}$. Now, use the definitions in eq. (23) coupled with the law of addition for cos so that eq. (24) becomes:

$$A\cos\phi + B\sin\phi = \sqrt{A^2 + B^2}\,(\cos\alpha\cos\phi + \sin\alpha\sin\phi) = \sqrt{A^2 + B^2}\,\cos(\phi - \alpha) = C\cos(\phi - \alpha) \tag{25}$$

We can always choose our orientation of coordinate axes in such a way as to make $\alpha$ zero, so that our expression finally becomes :

$$A\cos\phi + B\sin\phi = C\cos\phi \tag{26}$$

as long as C is properly chosen

Finally, we substitute eq. (26) into eq. (20) to obtain :

$$\mathbf{r = \frac{1}{C\cos\phi + GM/h^2}} \tag{27}$$

***Planetary Orbits***

Eq. (27) describes the nature of orbits in the gravitational field of a large, central, gravitating object. But what do these orbits look like?

If you have studied analytic geometry, you may recognize eq. (27) from your study of **conic sections**. As you may recall, conic sections are those geometric curves curves created from the intersection of a cone with a plane. If the plane is parallel to the base of the cone (or perpendicular to the axis of the cone), the resulting shape is a circle. If the plane is parallel to the axis (perpendicular to the base), the resulting curve is a parabola. Ellipses and hyperbolae are created by intersecting the cone at various angles. The general equations for these conic sections is

$$r = \frac{p}{1 + \epsilon \cos \phi} \tag{28}$$

where $\epsilon$ is the eccentricity of the ellipse, and p is (and this is one of my favorite terms in all of mathematics) the semi - lactus rectum of the ellipse. A course in advanced classical mechanics or celestial mechanics would go through a thorough derivation showing the relationship between the constants in eq. (28) with the constant in eq. (27). For the purposes of this classnote, I will simply posit the results and allow any one interested to speak with me or look up the more detailed derivation.

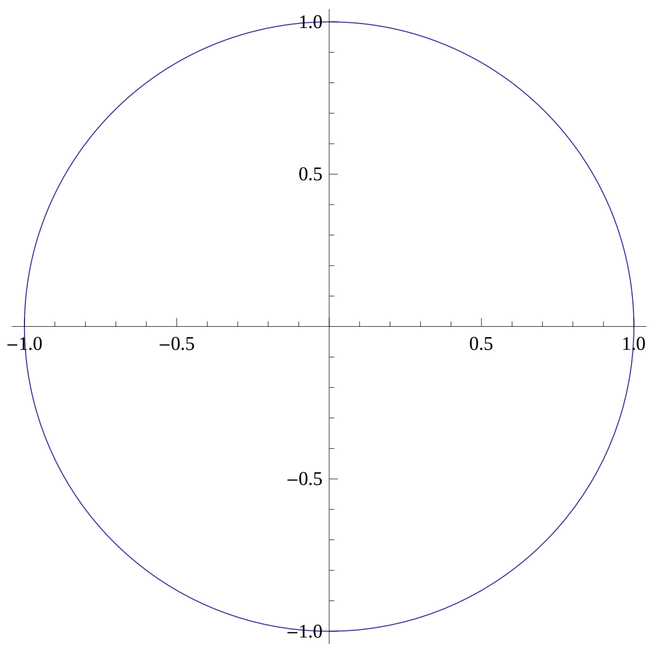
The shape of orbits predicted by eq. (28) depends critically on the value of eccentricity, the factor $\epsilon$ in the equation. Let' s look at each possible case and see the orbits we obtain :

***I. $\epsilon = 0$ : The Circle :***

If the eccentricity of the orbit is zero, the our equation becomes simply r = p, which we know to be the equation of a circle in polar coordinates. Not surprisingly, setting p = 1 we find :

In[49]:= `PolarPlot[1, {ϕ, 0, 2 π}]`

Out[49]=



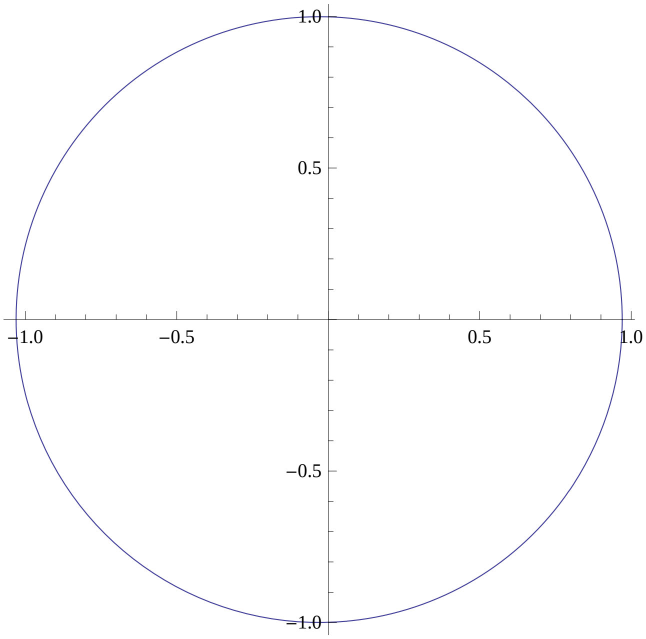

and in the process you learn how to do plots in polar coordinates.

### *II. $0 < \epsilon < 1$ : The Ellipse {$p = a\ (1 - e^2)$}*

Throughout human history, it was assumed that all planetary orbits were perfect circles. Even in Coipernicus' paradigm shifting work, De Revolutionibus Orbium Coelestium, he assumed the orbits of planets were circular in nature. It was not until the work of Johannes Kepler, several decades later, that we understood that planetary orbits were not circular, in fact, they are elliptical. (As a sign of my great admiration for Kepler, I named my first dog after him.) Orbits of most planets have small but non - zero eccentricites; the orbital eccentricity of the Earth is 0.03, meaning that our closest distance to the sun (the perihelion) is 3 % less than the average distance to the sun (semi - major axis, denoted by the variable "a")), and the greatest distance to the sun (aphelion) is 3 % greater than the semi - major axis. In solar system astronomy, we define the unit of length the astronomical unit to equal the semi - major axis of the Earth' s orbit. therefore, we can plot to scale the Earth' s orbit by setting $a = 1$ and $\epsilon = 0.03$ :

In[8]:=
```
Clear[a, ϵ, g1]
a = 1; ϵ = 0.03;
g1 = PolarPlot[a (1 - ϵ^2) / (1 + ϵ Cos[ϕ]), {ϕ, 0, 2 π}]
```

Out[10]=

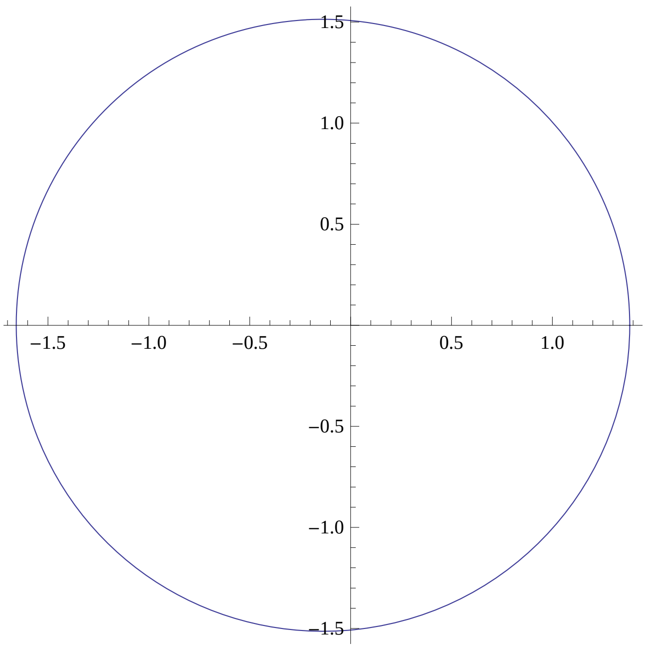



And that looks pretty circular.  If you look carefully, though, you will see that the ellipse intersects the + x axis at 0.97, and the - x axis at - 1.03, so that this curve is not perfectly circular.

We can see a more dramatic example of planetary elliptical orbits by plotting to scale the orbit of Mars, where a = 1.52 AU and $\epsilon = 0.09$ :

In[11]:=
```
Clear[a, ϵ, g2]
a = 1.52; ϵ = 0.09;
g2 = PolarPlot[a (1 - ϵ^2) / (1 + ϵ Cos[ϕ]), {ϕ, 0, 2 π}]
```

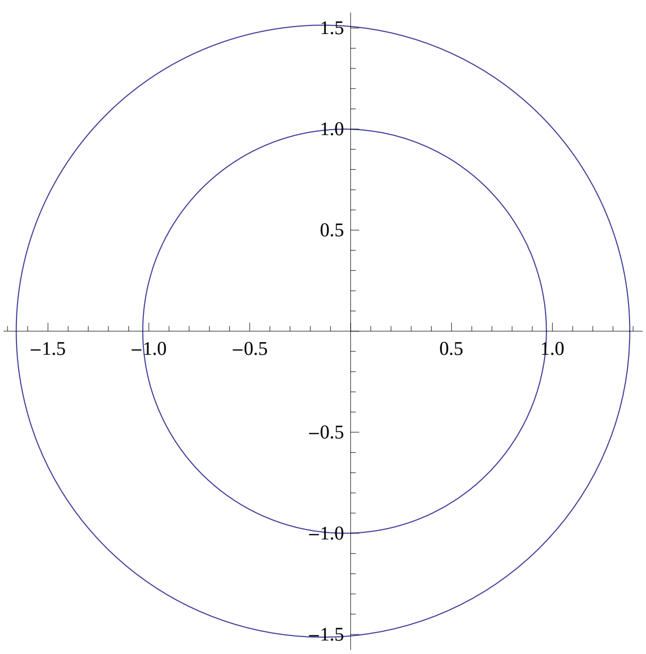
Out[13]=




We can superimpose these two graphs :

In[14]:=
```
Show[g1, g2]
```

Out[14]=



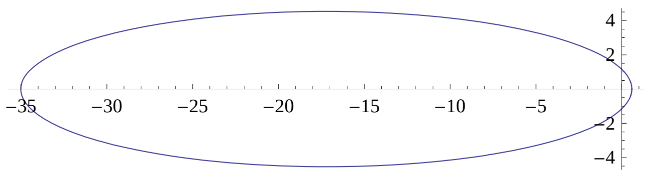
Because we have set 1 AU = 93 million miles, each tick mark along the x axis approximates 9.3 million miles. Notice along the + x axis that the orbits of Mars and Earth are only approx 35 million miles apart. We experience this configuration roughly every 17 years; Mars is particularly bright and noticeable during these times. When is our next such occurrence? Well, you might recall that in August 2010, Mars and Earth were at their closest in over 50, 000 years, so ... we have a bit of a wait to go for the next close opposition.

Comets, however, often have very elliptical orbits. The famous Comet Halley, with an orbit of 76 years, comes within approximately 0.5 AU of the sun, while its aphelion distance is out beyond Neptune' s orbit. This means Halley spends most of its life in the very cold reaches of the outer solar system, flashing relatively rapidly through the inner solar system once a lifetime. The last passage of Comet Halley near the Earth was in 1985 - 1986, just before most (if not all) of you were born. The next passage is in 2061. I wish you clear skies and good health as you observe this marvel of nature.

Using the parameters of its orbit (a = 17.8 AU, $\epsilon$ = 0.967) we plot the orbit of history' s most famous comet :

In[18]:=
```
Clear[a, ϵ]
a = 17.8; ϵ = 0.967;
g3 = PolarPlot[a (1 - ϵ^2) / (1 + ϵ Cos[ϕ]), {ϕ, 0, 2 π}]
```

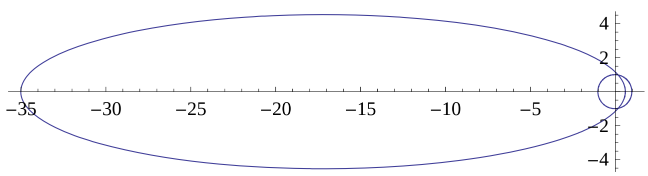
Out[20]=




Superimposing the Earth' s orbit with Halley' s :

In[21]:=
```
Show[g1, g3]
```

Out[21]=

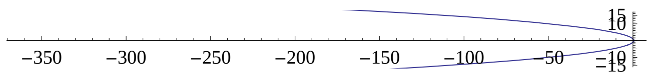



The small (almost) circle centered on the origin (i.e., location of the sun) is the Earth' s orbit. Perhaps this graphic helps explain why Halley is so rarely visible from Earth.

### *III. $\epsilon = 1$ : The parabola*

If the value of $\epsilon = 1$ exactly, the orbit is a parabola; as you know, parabolae are open ended curves, so that an object whose eccentricity is 1 never makes a complete orbit around the sun. The object comes in from a great distance away, swings around the sun, and then returns to its distant location. Arbitrarily setting p = 1 :

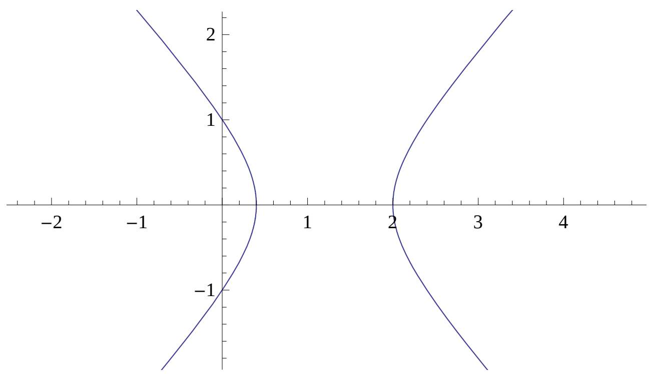
In[43]:=
```
Clear[ϵ]
ϵ = 1;
PolarPlot[1 / (1 + ϵ Cos[ϕ]), {ϕ, 0, 2 π}]
```

Out[45]=



### IV. $\epsilon > 1$ : The Hyperbola

If $\epsilon > 1$, the orbits obtained are also open. Again, setting p = 1 and $\epsilon = 1.5$ :

In[46]:=
```
Clear[ϵ]
ϵ = 1.5;
PolarPlot[1 / (1 + ϵ Cos[ϕ]), {ϕ, 0, 2 π}]
```

Out[48]=



### *Physical Signifcance of the eccentricities :*

We can see by plotting eq. (28) for various values of $\epsilon$ that we do in fact obtain each of the four conic sections (circle, ellipse, parabola, and hyperbola). But what is the connection between this geometry and physics?

To answer this, we will need to review some concepts you might have encountered in intro physics. If your course covered orbital motions in the chapters on gravity and/or circular motion, you may have learned about orbital and escape speed.

Orbital speed is the speed an object needs to achieve to continue moving around the Earth (or sun or any other central mass). Orbital speed is attained when the gravitational force on the object exactly equals the centripetal force on the orbiting object. Mathematically, this means :

$$\frac{m v^2}{r} = \frac{G M m}{r^2} \Rightarrow v_{orb} = \sqrt{G M / r}$$

where G is the Newtonian grav. cst, M and m are the masses of the central object and orbiting object, respectively, r is the distance from the center of the primary (central) mass, and $v_{orb}$ is the orbital velocity.

If an object has a speed less than $v_{orb}$, it will not be able to complete an orbit around the earth. $V_{orb}$ represents the minimum speed necessary to orbit the Earth, and if an object's speed is ***exactly*** equal to $v_{orb}$, then the object will orbit in a circular orbit with $\epsilon$=0.

If an object has a speed greater than $v_{orb}$ but less than the escape speed, then the object will execute an elliptical orbit and its eccentricity will lie between zero and one. The greater the orbital speed (as long as $v_{orb} < v_{esc}$), the greater the eccentricity of the orbit.

In order to achieve escape velocity, the kinetic energy of the object must equal its potential energy at a distance r from the center of the primary mass, in other words:

$$\frac{1}{2} m v^2 = \frac{G M m}{r} \Rightarrow v_{esc} = \sqrt{2 G M / r} = \sqrt{2}\, v_{orb} \tag{30}$$

Parabolic orbits occur when v = $v_{esc}$; hyperbolic orbits when v exceeds escape velocity.