As we begin our study of Fourier series, it is useful to learn some of the Mathematica functions that will allow us to analyze discontinuous functions and compute Fourier series.

**Plotting Piecewise Functions:**

Suppose we wish to consider the (by now) well known example:

\[
f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
1, & 0 < x < \pi 
\end{cases}
\]

We can plot it using the Which command:

```mathematica
Clear[f]
f = Which[-\[Pi] < x < 0, 0, 0 < x < \[Pi], 1];
Plot[f, {x, -\[Pi], \[Pi]}]
```

The Which command has the structure:

Which[test1, value1, test2, value2, ...]

Construct and plot the function defined by:

\[
f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
1, & 0 < x < \pi/2 \\
0, & \pi/2 < x < \pi 
\end{cases}
\]
Computing Fourier Series that are $2\pi$ periodic:

Let's consider again the function:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

We define it using the Which command:

```math
Clear[f]
f = Which[-\pi < x < 0, 0, 0 < x < \pi, 1]
```

We can compute the Fourier Series for this function using the `FourierTrigSeries` command:

```math
FourierTrigSeries[f, x, 5]
\frac{1}{2} + \frac{2 \sin[x]}{\pi} + \frac{2 \sin[3x]}{3\pi} + \frac{2 \sin[5x]}{5\pi}
```

The structure of this command computes the Fourier trig function series for the function $f$, in terms of the variable $x$, up to the fifth order terms. Let's see how this works for problem 11, p. 355 in Boas:

```math
Clear[g]
g = Which[-\pi < x < 0, 0, 0 < x < \pi, \sin[x]]; 
FourierTrigSeries[g, x, 6]
\frac{1}{\pi} - \frac{2 \cos[2x]}{3\pi} - \frac{2 \cos[4x]}{15\pi} - \frac{2 \cos[6x]}{35\pi} + \frac{\sin[x]}{2}
```

verifying the homework result.

The `FourierSeries` function produces the complex series:

```math
FourierSeries[g, x, 6]
\frac{1}{4} i e^{-ix} - \frac{1}{4} i e^{ix} + \frac{1}{\pi} - \frac{e^{-2ix}}{3\pi} - \frac{e^{2ix}}{3\pi} - \frac{e^{-4ix}}{15\pi} - \frac{e^{4ix}}{15\pi} - \frac{e^{-6ix}}{35\pi} - \frac{e^{6ix}}{35\pi}
```

Find the trig and complex series (up to terms of $\sin 10x$ and $\cos 10x$) for:

$$h(x) = \begin{cases} 0, & -\pi < x < 0 \\ -1, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$$
Computing Fourier series for functions that are 2L periodic:

In class we began considering functions that are periodic with some periodicity other than $2\pi$. Suppose we want to compute the 6th order Fourier trig series for the function:

$$f(x) = 2 - x, \quad -5 < x < 5$$

We make use of the following Mathematica command:

```
In[46]:= FourierTrigSeries[2 - x, x, 6, FourierParameters -> {1, \pi/5}]
```

$$2 - \frac{10 \sin\left(\frac{\pi x}{5}\right)}{\pi} + \frac{5 \sin\left(\frac{2\pi x}{5}\right)}{\pi} - \frac{10 \sin\left(\frac{3\pi x}{5}\right)}{3\pi} + \frac{5 \sin\left(\frac{4\pi x}{5}\right)}{2\pi} - \frac{2 \sin(\pi x)}{\pi} + \frac{5 \sin\left(\frac{6\pi x}{5}\right)}{3\pi}$$

(1)

Let's verify this by computing the Fourier series by direct integration. We know for a function that is 2L periodic, the Fourier series is written:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where the coefficients are computed from:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

In this case, $L = 5$ and $f(x) = 2 - x$, so we have:

```
In[48]:= a0 = Integrate[2 - x, {x, -5, 5}] / 5
Out[48]= 4
In[49]:= an = Integrate[(2 - x) Cos[n \pi x / 5], {x, -5, 5}] / 5
Out[49]= \frac{4 \sin[n \pi]}{n \pi}
In[50]:= bn = Integrate[(2 - x) Sin[n \pi x / 5], {x, -5, 5}] / 5
Out[50]= \frac{10 (n \pi \cos[n \pi] - \sin[n \pi])}{n^2 \pi^2}
```
We should be able to see immediate that the $a_n$ coefficients are zero since $\sin(n \pi) = 0$ for all integer values of $n$. The values of $b_n$ are:

$$b_n = \frac{10 \cos (n \pi)}{n \pi} = \frac{10 (-1)^n}{n \pi}$$

Therefore, the Fourier series becomes:

$$f(x) = \frac{4}{2} - \frac{10}{\pi} \left( \sin \left( \frac{\pi x}{5} \right) - \frac{\sin \left( \frac{2\pi x}{5} \right)}{2} + \frac{\sin \left( \frac{3\pi x}{5} \right)}{3} - \frac{\sin \left( \frac{4\pi x}{5} \right)}{4} + \ldots \right)$$

which is equivalent to eq. (1).