DISCONTINUOUS FUNCTIONS AND FOURIER SERIES IN MATHEMATICA

As we begin our study of Fourier series, it is useful to learn some of the Mathematica functions that will allow us to analyze discontinuous functions and compute Fourier series.

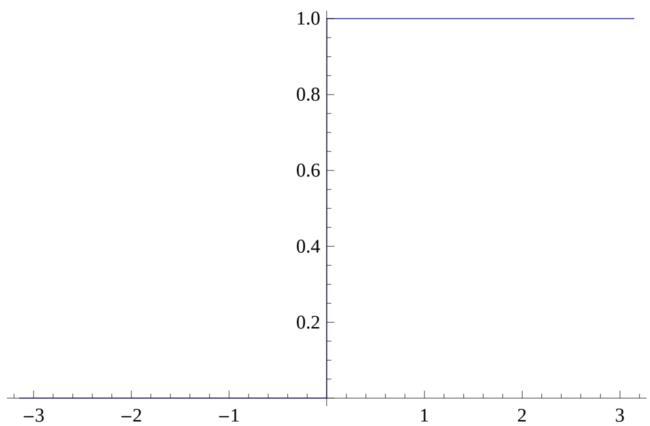
Plotting Piecewise Functions :

Suppose we wish to consider the (by now) well known example :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

We can plot it using the Which command :

```
Clear[f]
f = Which[-π < x < 0, 0, 0 < x < π, 1];
Plot[f, {x, -π, π}]
```



The Which command has the structure :

Which[test1, value1, test2, value2, ...]

Construct and plot the function defined by :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Computing Fourier Series that are 2 π periodic:

Let' s consider again the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

We define it using the Which command :

```
Clear[f]
f = Which[-π < x < 0, 0, 0 < x < π, 1]
```

We can compute the Fourier Series for this function using the FourierTrigSeries command :

```
FourierTrigSeries[f, x, 5]
```

$$\frac{1}{2} + \frac{2\,\mathrm{Sin}[x]}{\pi} + \frac{2\,\mathrm{Sin}[3\,x]}{3\,\pi} + \frac{2\,\mathrm{Sin}[5\,x]}{5\,\pi}$$

The structure of this command computes the Fourier trig function series for the function f, in terms of the variable x, up to the fifth order terms. Let' s see how this works for problem 11, p. 355 in Boas :

```
Clear[g]
g = Which[-π < x < 0, 0, 0 < x < π, Sin[x]];
FourierTrigSeries[g, x, 6]
```

$$\frac{1}{\pi} - \frac{2\,\mathrm{Cos}[2\,x]}{3\,\pi} - \frac{2\,\mathrm{Cos}[4\,x]}{15\,\pi} - \frac{2\,\mathrm{Cos}[6\,x]}{35\,\pi} + \frac{\mathrm{Sin}[x]}{2}$$

verifying the homework result.

The FourierSeries function produces the complex series :

```
FourierSeries[g, x, 6]
```

$$\frac{1}{4}\,i\,e^{-i\,x} - \frac{1}{4}\,i\,e^{i\,x} + \frac{1}{\pi} - \frac{e^{-2\,i\,x}}{3\,\pi} - \frac{e^{2\,i\,x}}{3\,\pi} - \frac{e^{-4\,i\,x}}{15\,\pi} - \frac{e^{4\,i\,x}}{15\,\pi} - \frac{e^{-6\,i\,x}}{35\,\pi} - \frac{e^{6\,i\,x}}{35\,\pi}$$

Find the trig and complex series (up to terms of sin 10 x and cos 10 x) for :

$$h(x) = \begin{cases} 0, & -\pi < x < 0 \\ -1, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$$

Computing Fourier series for functions that are 2 L periodic :

In class we began considering functions that are periodic with some periodicity other than 2π. Suppose we want to compute the 6 th order Fourier trig series for the function :

$$f(x) = 2 - x, \quad -5 < x < 5$$

We make use of the following Mathematica command :

In[46]:=
```
FourierTrigSeries[2 - x, x, 6, FourierParameters → {1, π / 5}]
```

$$2 - \frac{10\,\mathrm{Sin}\left[\frac{\pi x}{5}\right]}{\pi} + \frac{5\,\mathrm{Sin}\left[\frac{2\pi x}{5}\right]}{\pi} - \frac{10\,\mathrm{Sin}\left[\frac{3\pi x}{5}\right]}{3\pi} + \frac{5\,\mathrm{Sin}\left[\frac{4\pi x}{5}\right]}{2\pi} - \frac{2\,\mathrm{Sin}[\pi x]}{\pi} + \frac{5\,\mathrm{Sin}\left[\frac{6\pi x}{5}\right]}{3\pi} \tag{1}$$

Let' s verify this by computing the Fourier series by direct integration. We know for a function that is 2 L periodic, the Fourier series is written :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where the coefficients are computed from :

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

In this case, L = 5 and f (x) = 2 - x, so we have :

In[48]:=
```
a0 = Integrate[2 - x, {x, -5, 5}] / 5
```

Out[48]= 4

In[49]:=
```
an = Integrate[(2 - x) Cos[n π x / 5], {x, -5, 5}] / 5
```

Out[49]= $\frac{4\,\mathrm{Sin}[n\pi]}{n\pi}$

In[50]:=
```
bn = Integrate[(2 - x) Sin[n π x / 5], {x, -5, 5}] / 5
```

Out[50]= $\frac{10\,(n\pi\,\mathrm{Cos}[n\pi] - \mathrm{Sin}[n\pi])}{n^2\pi^2}$

We should be able to see immediate that the a_n coefficients are zero since sin(n π) =0 for all integer values of n. The values of b_n are:

$$b_n = \frac{10 \cos(n\pi)}{n\pi} = \frac{10 (-1)^n}{n\pi}$$

Therefore, the Fourier series becomes :

$$f(x) = \frac{4}{2} - \frac{10}{\pi}\left(\sin\left(\frac{\pi x}{5}\right) - \frac{\sin\left(\frac{2\pi x}{5}\right)}{2} + \frac{\sin\left(\frac{3\pi x}{5}\right)}{3} - \frac{\sin\left(\frac{4\pi x}{5}\right)}{4} + \ldots\right)$$

which is equivalent to eq. (1).