FOURIER SERIES ON ANY INTERVAL

Overview

We have spent considerable time learning how to compute Fourier series for functions that have a period of 2π on the interval $(-\pi,\pi)$. We have also seen how Fourier series generate some very interesting results, but you might have wondered how general these results are. After all, there is a lot of physics that involves functions that are not 2π periodic on $(-\pi,\pi)$; does Fourier analysis work for them? This classnote is designed to illustrate some of the main principles developed in sections 8 and 9 of Boas (Ch. 7) showing how to generalize Fourier analysis to any "reasonably well behaved function."

The last section of this classnote deals with the material in Boas' section 9. I think this section (of this classnote) is particularly important since I don't think Boas sufficiently stressed the significance of this material.

Fourier Series on Other Intervals

So far we have considered functions only on the interval $(-\pi,\pi)$. There is no reason we could not have chosen another interval; we just used $(-\pi,\pi)$ as a starting point. As your text notes, though, it is important to understand both the function and its interval in analyzing it; for instance, if we consider the simple function $f(x) = x^2$ on the two intervals $(-\pi, \pi)$ and $(0,2\pi)$, we will get different Fourier series for them. First, let's plot several cycles of the function on the interval $(-\pi,\pi)$:

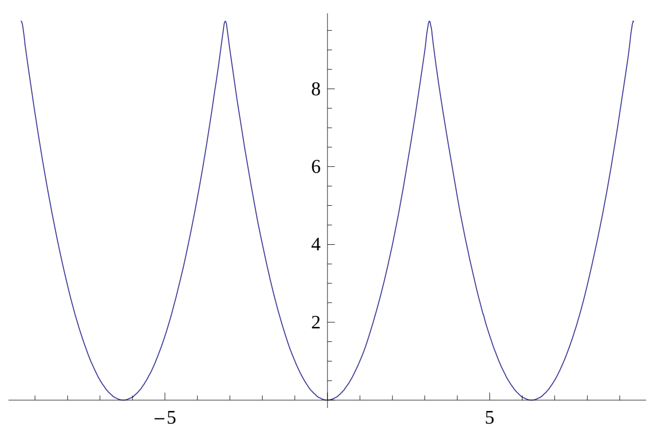


Fig. 1 Plot of $y = x^2$ defined on $(-\pi, \pi)$

As we would expect, the function is even on this interval, and if we calculate the Fourier series for this function, we find :

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n x)}{n^2} \tag{1}$$

If we translate this function by π, our function is now defined on the interval $(0,2\pi)$. We can still write a Fourier series for this function in familiar terms:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n x) + b_n \sin(n x) \tag{2}$$

The only difference now is that our limits of integration for computing coefficients are also shifted :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx;\ \ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(n x)\,dx;\ \ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(n x)\,dx \tag{3}$$

Make sure you notice that we are now integrating between 0 and 2π.

We can compute these coefficients easily :

```
In[238]:= Clear[f]
          f[x_] := x^2
          Integrate[{f[x], Cos[n x] f[x], Sin[n x] f[x]},
           {x, 0, 2 π}, Assumptions → Element[n, Integers]]
```

Out[240]= $\left\{\frac{8\pi^3}{3}, \frac{4\pi}{n^2}, -\frac{4\pi^2}{n}\right\}$

We observe that the b_n are non zero as they were on the interval $(-\pi,\pi)$. The reason for this becomes clear when we plot several cycles of $y = x^2$defined on $(0,2\pi)$:

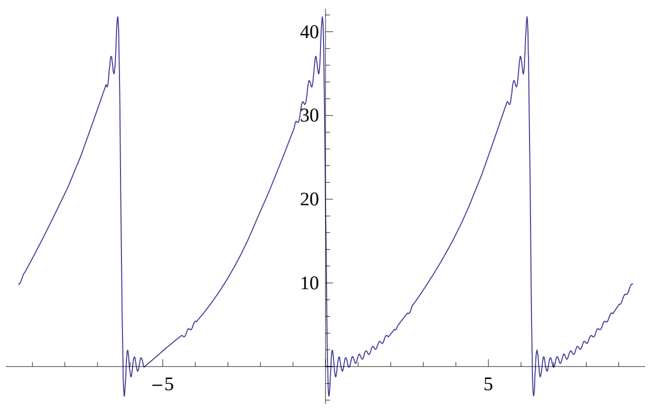


Fig. 2 A graph of $y = x^2$ defined on $(0, 2\pi)$

Unlike the Fourier series in equation (1) which involves only cos terms (i.e., even terms) because the function is even, the Fourier series defined on $(0,2\pi)$ involves both cos and sin terms since the function is neither even nor odd when defined and graphed on this interval. We can use the coefficients computed immediately above and write the Fourier series for this interval as:

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4\pi\cos(n x)}{n^2} - \frac{4\pi^2\sin(n x)}{n}\right) \tag{4}$$

This exercise shows that we can compute Fourier series for other intervals, but that we have to be careful to recompute the coefficients

Fourier Series with different Periodicities

Suppose we have a function that is periodic on the interval (-1, 1), or some other interval not involving simple multiples of π. The extension of Fourier series to such instances is quite simple. Suppose again we are dealing with our function of $y = x^2$; this time we wish to consider it its behavior if it is defined on the interval (-1,1). In this case, its period is 2. How can we define its Fourier series for this interval?

For the more general case of a function whose periodicity is *2l*, we can show that both sin($\frac{n\pi x}{l}$) and cos($\frac{n\pi x}{l}$) are *2l* periodic. We know that a function is periodic with period *p* if f(x+p)=f(x). In this case, we can show that sin($\frac{n\pi x}{l}$) is *2l* periodic because:

$$\sin\left[\frac{n\,\pi}{l}(x+2\,l)\right]=\sin\left(\frac{n\,\pi\,x}{l}+2\,n\pi\right)=\sin\left(\frac{n\,\pi\,x}{l}\right)\cos(2\,n\pi)+\sin(2\,n\,\pi)\cos\left(\frac{n\,\pi\,x}{l}\right) \tag{5}$$

We know that cos(2 n π) is equal to 1 for all integer values of n, and sin (2 n π) is zero for all integers, so that the expression in eq. (5) reduces to:

$$\sin\left[\frac{n\,\pi}{l}(x+2\,l)\right]=\sin\left(\frac{n\,\pi\,x}{l}\right) \tag{6}$$

and we have shown that sin($\frac{n\pi x}{l}$) is in fact *2l* periodic. A similar analysis will show that cos($\frac{n\pi x}{l}$) is also *2l* periodic.

It is a simple enough matter to write the general form of the Fourier series for any periodic interval *L*. The general form of the series is:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\,\pi\,x}{L}\right)+b_n\sin\left(\frac{n\,\pi\,x}{L}\right)\right] \tag{7}$$

and our coefficients are defined as :

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx \tag{8}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\,\pi\,x}{L}\right)dx \tag{9}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\,\pi\,x}{L}\right)dx \tag{10}$$

Notice that I am using upper case "L" to avoid confusion between "l" and "1". More substantively, notice that if the length of the interval is 2 π, then L=π, and the definitions of the coefficients and of the Fourier series reduce to the familiar forms we know for functions on the interval (-π,π).

Let's try now to compute the Fourier series for $y = x^2$on the interval (-1,1). Here, L=1, so we have:

In[241]:=
```
a0 = 1/1 Integrate[x^2, {x, -1, 1}]
```

Out[241]= $\frac{2}{3}$

So the first term in the Fourier series will be 1/2 of this, or 1/3.

In[242]:=
```
an = Integrate[x^2 Cos[(n π x) / 1], {x, -1, 1}, Assumptions → Element[n, Integers]]
```

Out[242]= $\frac{4\,(-1)^n}{n^2\,\pi^2}$

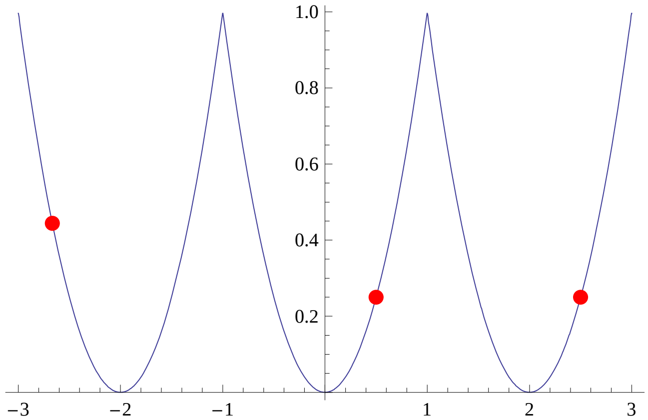
Where I do not explicitly write the factor of "1" before the integral.

In[243]:=
```
bn =  Integrate[x^2 Sin[(n π x) / 1], {x, -1, 1}, Assumptions → Element[n, Integers]]
```

Out[243]= 0

Not surprisingly, the b coefficients are zero. We could have predicted this since our function is even on this interval, and we expect only even terms, thus no odd (sin) terms. Verifying this Fourier series by plotting three cycles of its sum over 100 terms:

In[251]:=
```
Plot[1 / 3 + (4 / π^2) Sum[(-1) ^n Cos[(n π x) / 1] / n^2, {n, 1, 100}],
 {x, -3, 3}, Epilog → {Red, PointSize[Large],
   Point[{{1 / 2, (1 / 2) ^2}, {2 + 1 / 2, (1 / 2) ^2}, {-2 - 2 / 3, (-2 / 3) ^2}}]}]
```

Out[251]=


The obvious red dots verify that each point lies on the curve defined by $y = x^2$over the interval (-1,1)

■ One more example

As a final example, let' s work out problem 15 a) on p. 363 of Boas. We are asked to find the Fourier series for the function defined by :

$$f\,(x) = x,\ \ -1 \leq x < 1 \tag{11}$$

Our interval is 2, so L = 1. Since our function is clearly odd over this interval, we know we need to compute only the b_n terms:

In[247]:=
```
bn = 1/1 Integrate[x Sin[(n π x) / 1], {x, -1, 1}, Assumptions → Element[n, Integers]]
```

Out[247]= $-\frac{2\,(-1)^n}{n\,\pi}$

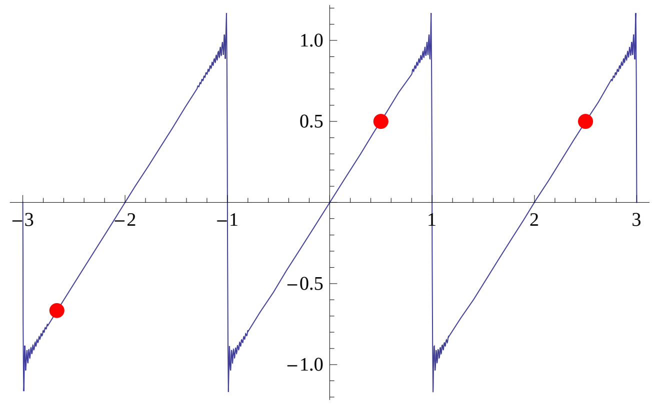
And our Fourier series is :

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin(n\pi x)}{n} = \frac{-2}{\pi}\left[-\sin(\pi x) + \frac{\sin(2\pi x)}{2} - \frac{\sin(3\pi x)}{3} + \ldots\right] \tag{12}$$

Plotting three cycles of this function for verification :

```
In[249]:= Plot[-2/π Sum[(-1)^n Sin[n π x]/n, {n, 1, 100}], {x, -3, 3},
  Epilog → {Red, PointSize[Large], Point[{{1/2, 1/2}, {2+1/2, 1/2}, {-2-2/3, -2/3}}]}]
```

Out[249]=



And the bright beacons of Mathematica tell us we have chosen our coefficients well.

Functions that are not periodic

The previous sections have shown us how to compute the Fourier series for functions that are periodic on some interval other than (-*π*,*π*). What do we do if we want to use the Fourier series for a function that is not periodic on any interval? Suppose we need to find the Fourier series of a function that is defined on an interval (0,L)? (We will encounter such situations in Ch. 13; some of you have already encountered these in Ch. 3 of Griffiths.) This is the essential question posed in Sect. 9 of Boas.

So, suppose we need to find the Fourier series for the function f(x)=x defined on the interval (0,2). How can we do this since this function is not periodic?

Simple: we make it periodic. Consider one possible way of doing this. The graph of our original function is:

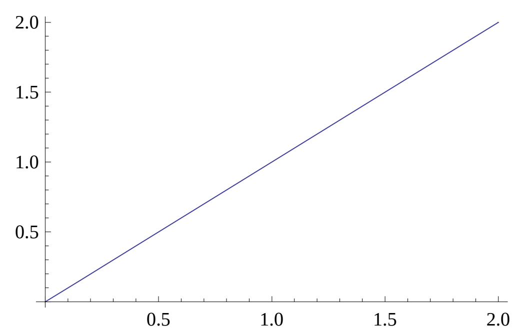


Fig. 3. Graph of y = x on (0, 2)

Now, we can make this periodic by extending the graph in the negative half plane; for instance, we can make this an even function :

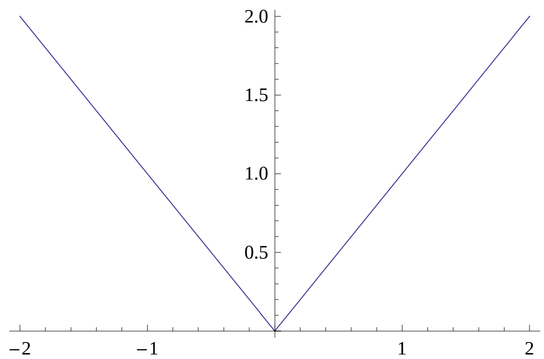


Fig. 4. Graph of y = Abs[x] on (-2, 2)

Or we can make this an odd function :

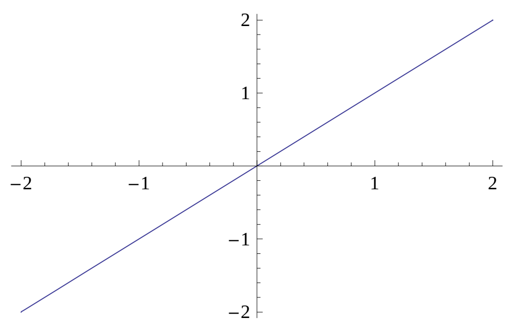


Fig. 5. Graph of y = x on (-2, 2)

You know how to find easily the Fourier series for the functions defined in Figs. 4 and 5. For instance, for the function defined as in Fig .4, the Fourier series on the interval (-2, 2) is :

$$f\,(x) = 1 - \frac{8}{\pi^2}\left[\sum_{n=1,3,5}^{\infty} \frac{\cos\left(\frac{n\,\pi\,x}{2}\right)}{n^2}\right] \tag{13}$$

Not surprisingly, the even extension of the function into the left half plane produces a Fourier series that consists of only cos (even) terms. The graph of this series is:

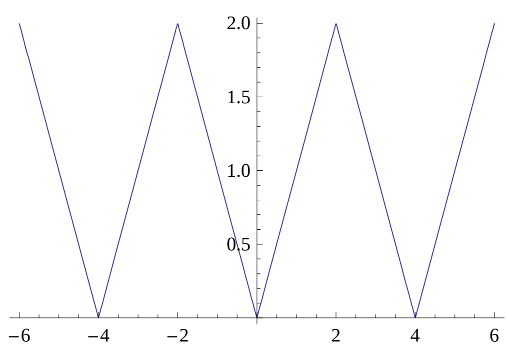


Fig. 6. Fourier series of y = Abs[x] on (-6, 6)

We can just as easily find the Fourier series for the odd function described by the graph in Fig. 5. Since this is an odd function, we expect to find only sin terms, and the computed Fourier series for this function is :

$$f(x) = \frac{-4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \mathrm{Sin}\left[\frac{n\pi x}{2}\right]}{n} \tag{14}$$

and yields the graph :

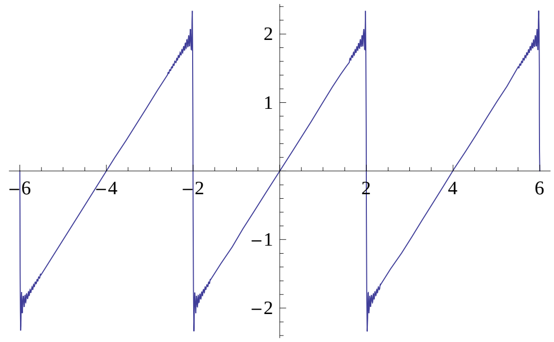


Fig. 7. Fourier series for y = x on (-6, 6)

As we see from the preceding discussion, we can take any function and make it periodic in order to find its Fourier series.

The key point is that we are interested only in the Fourier series that defines our region of interest. So if we only care about the behavior of the function on (0, 2), we can still construct the Fourier series assuming the function is periodic on (-2, 2), and just use that portion of the Fourier series that applies to (0, 2). Notice that theFourier series, eqs. (13) and (14), have very different behaviors (one is even, one is odd; one varies as 1/n, one as $1/n^2$), yet they give us exactly the same behavior on the interval (0,2).

So how do we know whether to extend the function to make it an even or an odd function? Sometimes it doesn't matter; sometimes the symmetry of the problem will make it clear that we must use either the even or odd extension.

In fact, we have an infinite array of ways we can make a function periodic. For instance we could have taken our initial function (y=x on (0,2)) and defined it on (-2,2) as:

$$f(x) = \begin{cases} x, & 0 \le x < 2 \\ 0, & -2 \le x < 0 \end{cases}$$

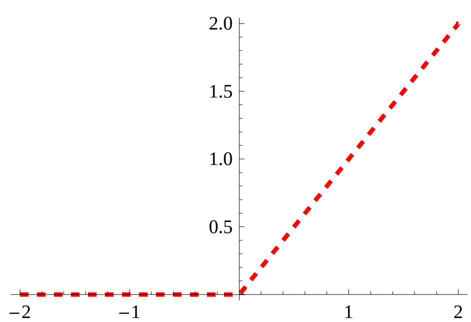


Fig. 8. Plot of function in (15) on (-2, 2)

(I plot the function with red dashes so you can distinguish the portion of the graph on (-2,0) from the negative x axis).

If we compute the Fourier series for this series, we get a much more complicated expression :

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{\cos\left(\frac{n\pi x}{2}\right)}{n^2} - \frac{2}{n} \sum_{n=1}^{\infty} \frac{(-1)^n \sin\left(\frac{n\pi x}{2}\right)}{n} \tag{15}$$

which we might expect since the function defined immediately above is neither even nor odd, so our Fourier series includes both sin and cos terms. Yet, this Fourier series produces exactly the same behavior on the interval (0, 2) :

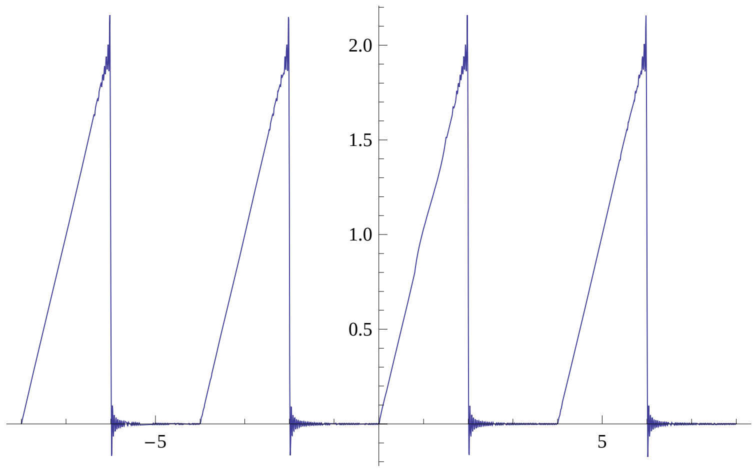


Fig. 9. Graph of Fourier series (15) on (-8, 8)

So why not choose this formulation of y = x on (0, 2)? We could, but why make our lives more difficult than necessary. Assuming even or odd symmetry makes our Fourier functions much simpler (and the calculation of those series much easier). Of course, as noted before, sometimes the constraints of the problem will make it clear to us whether we should make use of the even or odd extension, but we can worry about those details in Chapter 13.