Fourier Series on other Intervals : Notes from Class

In class on W we investigated Fourier series on intervals other than \((-\pi, \pi)\). For functions that are 2L periodic (where 2L is the wavelength), we compute Fourier coefficients and the Fourier series making the modifications derived in class:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x}{L} \right) + b_n \sin \left( \frac{n \pi x}{L} \right)
\]

where:

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx
\]

Example 1: Consider \(f(x) = x^2\) on \((-2, 2)\):

It is important to recall that 2L is the length of the interval, therefore 2L = 2 - (-2) so that the value of L to use in the calculations is L = 2.

\[
a_0 = \frac{1}{2} \int_{-2}^{2} x^2 \, dx \quad a_n = \frac{1}{2} \int_{-2}^{2} x^2 \cos \left( \frac{n \pi x}{2} \right) \, dx \quad b_n = \frac{1}{2} \int_{-2}^{2} x^2 \sin \left( \frac{n \pi x}{2} \right) \, dx
\]

Now, since \(f(x)\) is even on \((-2, 2)\) we can employ symmetry to write:

\[
a_0 = \frac{2}{2} \int_{0}^{2} x^2 \, dx \quad a_n = \frac{2}{2} \int_{0}^{2} x^2 \cos \left( \frac{n \pi x}{2} \right) \, dx \quad b_n = 0
\]

Now, going the lazy route and letting Mathematica do the heavy lifting:

\[
\text{In}[333]:= \text{a0} = \text{Integrate}[x^2, \{x, 0, 2\}]
\]

\[
\text{Out}[333]= \frac{8}{3}
\]

\[
\text{In}[332]:= \text{an} = \text{Integrate}[x^2 \cos[n \pi x / 2], \{x, 0, 2\}]
\]

\[
\text{Out}[332]= \frac{8}{n^3 \pi^3} \left(2 n \pi \cos[n \pi] + \left(-2 + n^2 \pi^2\right) \sin[n \pi]\right)
\]

and let's just be sure:

\[
\text{In}[337]:= \text{bn} = \text{Integrate}[x^2 \sin[n \pi x / 2], \{x, -2, 2\}]
\]

\[
\text{Out}[337]= 0
\]

It is easy to show that the \(a\) coefficients reduce to:

\[
a_n = \frac{16}{n^2 \pi^2} (-1)^n
\]

so that our Fourier series is:

\[
f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n \pi x / 2)}{n^2}
\]
Verifying through Mathematica:

\[
\text{In[339]:= } \text{Plot} \left( \frac{4}{3} + \frac{16}{\pi^2} \sum (-1)^n \cos \left( \frac{n \pi x}{2} \right) \right)_{n=1}^{21}, (x, -6, 6)
\]

\[
\text{Out[339]=}
\]

Note that the values are consistent with our function \((f(2) = 4)\) and the function repeats with a periodicity of 4.

Now, let’s examine the same function on \((0, 4)\). Since our limits are not symmetric across the origin, we cannot make use of symmetry arguments. Our integrals become:

\[
a_0 = \frac{1}{2} \int_0^4 x^2 \, dx \quad a_n = \frac{1}{2} \int_0^4 x^2 \cos \left( \frac{n \pi x}{2} \right) \, dx \quad b_n = \frac{1}{2} \int_0^4 x^2 \sin \left( \frac{n \pi x}{2} \right) \, dx
\]

Even though the limits have changed, \(L = 2\) since the total length of the interval, \(2L = 4\).

Computing coefficients:

\[
\text{Clear}[a_0, a_n, b_n, L, f]
\]

\[
\text{In[340]= } L = 2;
\]

\*(I know I have stressed we should not use capital letters for variables; but I know that "L" is not restricted for Mathematica use, so let’s use it. *)

\[
f[x_] := x^2
\]

\[
a_0 = \left( \frac{1}{L} \right) \text{Integrate}[f[x], \{x, 0, 4\}]
\]

\[
a_n = \left( \frac{1}{L} \right) \text{Integrate}[f[x] \cos \left( \frac{n \pi x}{L} \right), \{x, 0, 4\}]
\]

\[
b_n = \left( \frac{1}{L} \right) \text{Integrate}[f[x] \sin \left( \frac{n \pi x}{L} \right), \{x, 0, 4\}]
\]

\[
\text{Out[342]= } \frac{32}{3}
\]

\[
\text{Out[343]= } \frac{8 \left( 2 n \pi \cos \left( 2 n \pi \right) + \left( -1 + 2 n^2 \pi^2 \right) \sin \left( 2 n \pi \right) \right)}{n^3 \pi^3}
\]

\[
\text{Out[344]= } -\frac{8 \left( 1 + \left( -1 + 2 n^2 \pi^2 \right) \cos \left( 2 n \pi \right) - 2 n \pi \sin \left( 2 n \pi \right) \right)}{n^3 \pi^3}
\]

And we get our three outputs. Knowing that \(\sin \left( 2 n \pi \right)\) is always zero for integer values of \(n\), these complicated expressions reduce quite nicely, and we have:

\[
a_n = \frac{16}{n^2 \pi^2} \quad b_n = -\frac{16}{n \pi}
\]
Our Fourier series then becomes:

\[
    f(x) = \frac{32}{6} + \sum_{n=1}^{\infty} \frac{16}{n^2} \cos\left(\frac{n \pi x}{2}\right) \frac{\pi}{n^2} + \sum_{n=1}^{\infty} \frac{-16}{\pi} \frac{\sin\left(\frac{n \pi x}{2}\right)}{n^2}.
\]

Plotting three cycles of this function using the first 31 terms of the expansion:

\[
    \text{Plot}\left[\frac{32}{6} + \sum\left(\frac{16}{\pi^2} \cos\left(\frac{n \pi x}{2}\right) \frac{\pi}{n^2} + \frac{-16}{\pi} \frac{\sin\left(\frac{n \pi x}{2}\right)}{n^2}\right), \{n, 1, 31\}, \{x, -4, 8\}\right]
\]

And we reproduce our function over three cycles.