

# NOTES ON THE GRADIENT

In this discussion, we investigate properties of the gradient and in the process learn several important and useful Mathematical functions and techniques.

## *Simple examples of the gradient of a scalar field*

Let's start by considering the temperature in room that has a fireplace (or some other heating source) in one part of the room and some open windows scattered around the room. We will learn in Chapter 13 (partial differential equations) how we can solve such problems analytically, but for now, we just need to focus on the fact that the temperature in the room will vary from one place to another. In other words, the temperature in the room will depend on its  $(x, y, z)$  coordinates. We can express this as:

$$T = T(x, y, z)$$

This is a scalar field since temperature is a scalar quantity. Imagine now a very temperature sensitive (and slow moving) fly that is moving through the room. When the fly will measure some temperature when it is at an initial position  $(x_1, y_1, z_1)$ . As the fly moves to another position in the room labeled as  $(x_2, y_2, z_2)$ , the fly will measure a change in temperature. This change in temperature will occur because the fly has changed positions in  $x, y,$  and  $z,$  and also because the temperature changes from one position in the room to another. You learned in multivariable calculus that you can describe the total change in temperature between two points close to each other,  $dT,$  by:

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \quad (1)$$

Now, let's examine eq. (1) more closely. The right - hand side consists of a sum of three products; this pattern is becoming familiar to us. The right - hand side of eq. (1) is reminiscent of a dot product of two vectors. The two vectors that produce the scalar product on the right side of eq. (1) are :

$$\left( \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \right) \text{ and } (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) \quad (2)$$

or that the total change in temperature between two positions is :

$$dT = \left( \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \right) \cdot (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) \quad (3)$$

Let's see what each of the terms on the right side of eq. (3) mean. We can understand the term in the second parentheses by recalling the definition of the position vector, i.e., the vector from the origin to a point in  $(x, y, z)$  space:

$$\mathbf{r} = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}} \quad (4)$$

Therefore, the vector representing an increment of displacement from  $(x, y, z)$  to  $(x + dx, y + dy, z + dz)$  is written as :

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}} \quad (5)$$

this is called the infinitesimal displacement vector and the symbol  $d\mathbf{l}$  is often reserved to describe this vector. Therefore, we see that the second term in eq. (3) is just an increment of displacement along a line from the origin to our point of measurement.

The first term on the right side of eq. (3) is the **gradient of the temperature field** and is represented by the symbol  $\nabla T$ , so we can write  $dT$  from eq. (3) as:

$$dT = \nabla T \cdot d\mathbf{l} \quad (6)$$

We can recast eq. (6) using the basic definition of the dot product :

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T| |d\mathbf{l}| \cos \theta \quad (7)$$

Now, if we set the magnitude of  $d\mathbf{l}$  to some specific value, say for specificity one, then the largest value of  $dT$  will occur when  $\theta=0$  so that  $\cos \theta = 1$ ; this means that the largest value of  $dT$  will occur when  $\nabla T$  and  $d\mathbf{l}$  lie in the same direction. This allows us to see that the direction of the gradient vector represents the direction of the maximum change of the parameter we are studying (in this case, the temperature of the room). Additionally, the magnitude of the gradient,  $|\nabla T|$  is the rate of change of  $T$  in the direction of maximum change.

**Ex. 1:** Suppose the function that describes the temperature at any point in the room is given by:

$$T(x, y, z) = 100 \cos\left(\frac{x}{10}\right) \sin\left(\frac{y}{10}\right) \cos z \quad (8)$$

then the gradient of  $T$  is :

$$\begin{aligned} \nabla T = \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} = & -10 \sin\left(\frac{x}{10}\right) \sin\left(\frac{y}{10}\right) \cos z \hat{\mathbf{x}} + \\ & 10 \cos\left(\frac{x}{10}\right) \cos\left(\frac{y}{10}\right) \cos z \hat{\mathbf{y}} - 100 \cos\left(\frac{x}{10}\right) \sin\left(\frac{y}{10}\right) \sin z \hat{\mathbf{z}} \end{aligned} \quad (9)$$

Eq. (9) tells us that if we are at the point in the room  $(10\pi, 10\pi, \pi)$ , that the direction of greatest change in temperature is :

$$\nabla T = 0 \hat{\mathbf{x}} - 10 \hat{\mathbf{y}} + 0 \hat{\mathbf{z}} \quad (10)$$

or in the minus  $y$  direction, and that the rate of change of temperature at this point is the magnitude of the gradient, or

$$|\nabla T| = \sqrt{(-10)^2} = 10 \quad (11)$$

**Ex. 2:** Suppose the height of a hill,  $z(x, y)$ , is given by :

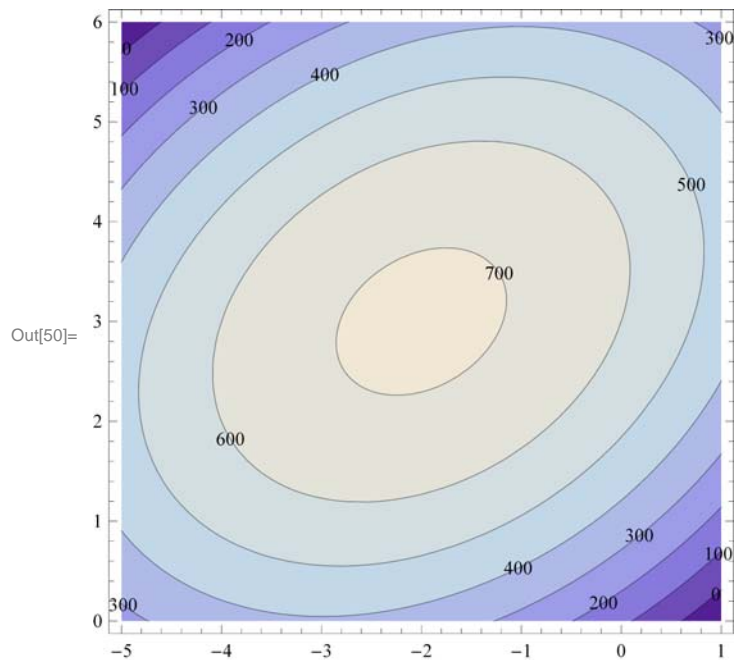
$$z(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12) \quad (12)$$

where  $x$  and  $y$  are the distance in some appropriate units measured from an origin of coordinates.

### Using the ContourPlot function in Mathematica

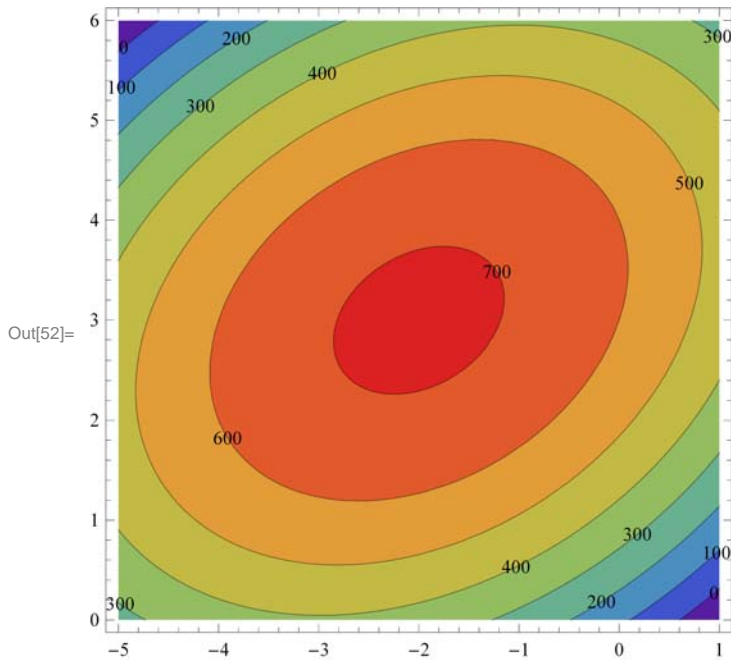
We can use the ContourPlot feature of Mathematica to draw the contour lines of this topography; in the plot below, each curve represents a certain, constant height :

```
In[48]:= Clear[z]
z = 10 (2 x y - 3 x^2 - 4 y^2 - 18 x + 28 y + 12);
ContourPlot[z, {x, -5, 1}, {y, 0, 6}, ContourLabels -> All]
```



Look for a moment at the structure of the ContourPlot command. Note the capitalization rules that should be becoming familiar to you. The ContourPlot function call requires the use of brackets; inside the brackets you put the function to be plotted, and the ranges in x and y over which you will plot the function. So here, we plot z between  $-5 < x < 1$  and  $0 < y < 6$ . The final statement "ContourLabels->All" instructs Mathematica to label each contour line. And if you don't like the splash of various shades of purple, you can modify the color scheme used in the display (my 12 yr old daughter highly recommends the rainbow color function shown below:)

```
In[52]:= ContourPlot[z, {x, -5, 1}, {y, 0, 6}, ContourLabels -> All, ColorFunction -> "Rainbow"]
```



In the plot above, we can see that the shape of the hill is ellipsoidal, oriented SW to NE (assuming N is at the top of the plot). Each elliptical curve represents a certain height; the inner most closed loop for instance shows every where the hill's height is 700 units

### ***The height and location of the top of the hill; the Mathematica "Solve" Function***

Can we find the location of the top of the hill and also the value of the height at this point. First, we know that the hill achieves maximum elevation when  $dz = 0$ . This means that when we are at the very top of the hill (at a maximum), the value of all derivatives there is zero. But in this case,  $z$  is a function of  $x$  and  $y$ , so the condition that  $dh = 0$  means that we want to find where  $\partial z/\partial x$  and  $\partial z/\partial y$  are simultaneously zero. We take partial derivatives of eq.(12) :

$$\frac{\partial z}{\partial x} = 2y - 6x - 18 = 0$$

$$\frac{\partial z}{\partial y} = 2x - 8y + 28 = 0$$
(13)

We can use elementary methods of algebra to find the  $(x, y)$  location that satisfies these equations simultaneously, and this will be the  $(x, y)$  location of the top of the hill. We can also use the Solve feature in Mathematica :

```
In[15]:= Solve[{2 y - 6 x - 18 == 0, 2 x - 8 y + 28 == 0}, {x, y}]
```

```
Out[15]= {{x -> -2, y -> 3}}
```

Since we are solving two equations simultaneously, we write them inside braces since Mathematica regards this as a list of equations. Notice that we use the double equal sign "==". Remember that Mathematica has 3 different types of equal signs; the "traditional" equal sign "=" is used when you are assigning a value to a variable. For instance, if you want the variable p to have the numerical value 12 throughout the calculation, you would use the single equal sign and assign  $p = 12$ . However, in equations, you are solving for the values of the variables that satisfy this particular set of equations; in Mathematica this is indicated by use of the "=" sign. Finally, notice that we have to declare the variables we are solving for; it is obvious to you that we want to solve for x and y, however, this needs to be input explicitly in the format shown : "{x,y}".

Output line [15] tells us that the solution to these equations are  $x = -2$  and  $y = 3$ , so that the top of the hill lies at (-2, 3). If you refer back to the contour plot, you will be able to verify this result.

### **Mathematica Alert: The "slash-dot" command**

Now, we can make use of another Mathematica feature to evaluate the function z at (-2, 3) to determine the value of the maximum height. We make use of the "slash-dot" feature :

```
In[22]:= z /. {x -> -2, y -> 3}
```

```
Out[22]= 720
```

The syntax in input line [22] means "in z, substitute  $x=-2$  and  $y=3$  and evaluate". We use braces again since we are assigning variables to two variables. In the units chosen, the maximum height of the hill is 720.

As another example of "/", suppose we want to evaluate a polynomial at a specific value of x. If our function is  $y = x^5 - 4x^4 + 3x^3 - 2x^2 + x - 3$  to be evaluated at  $x=5$ , we write:

```
In[23]:= Clear[x, y]
y = x^5 - 4 x^4 + 3 x^3 - 2 x^2 + x - 3;
y /. x -> 5
```

```
Out[25]= 952
```

If we want to evaluate y at every integer value of x between 0 and 8, we write :

```
In[26]:= y /. x -> {0, 1, 2, 3, 4, 5, 6, 7, 8}
```

```
Out[26]= {-3, -4, -17, -18, 161, 952, 3171, 8138, 17797}
```

And the output is presented as a list of evaluations.

### **Vector Analysis with Mathematica**

Let's finish by making use of the Mathematica vector analysis package. First we need to load this package :

```
In[28]:= Needs["VectorAnalysis`"]
```

Remember to enter "shift-enter" to load the vector analysis package. Let's find the gradient of the function z(x, y) from eq. (12). I will change the variable name from z(x,y) to h to avoid any confusion with the use of z as a Cartesian coordinate. Making use of good programming technique and clearing all variables, we get :

```
In[38]:= Clear[h, x, y, z]
         h = 10 (2 x y - 3 x^2 - 4 y^2 - 18 x + 28 y + 12);
         Grad[h, Cartesian[x, y, z]]
```

```
Out[40]= {10 (-18 - 6 x + 2 y), 10 (28 + 2 x - 8 y), 0}
```

### ***Mathematica Alert: Components of a vector and the "%" command***

Since the gradient is a vector, the output shows the components of the gradient as elements in a list. Notice that the elements of the gradient vector in output line [40] are the same (apart from the multiplicative factor of 10) from our results in eqs. (13). We can use the results in output line [40] to determine directly the coordinates of the top of the hill. We use the following "Solve" statement :

```
In[41]:= Solve[{{%[[1]] == 0, %[2]] == 0}, {x, y}]
```

```
Out[41]= {{x -> -2, y -> 3}}
```

We recognize that input line [41] is solving two equations in the variables x and y; and we recognize output line [41] as the same solution we determined earlier. The Solve statement above introduces us to two new Mathematica properties. We know we want to take each expression for the partial derivative of h and set it equal to zero. Remember that the gradient is a vector, so the results in output line [40] are the components of a vector.

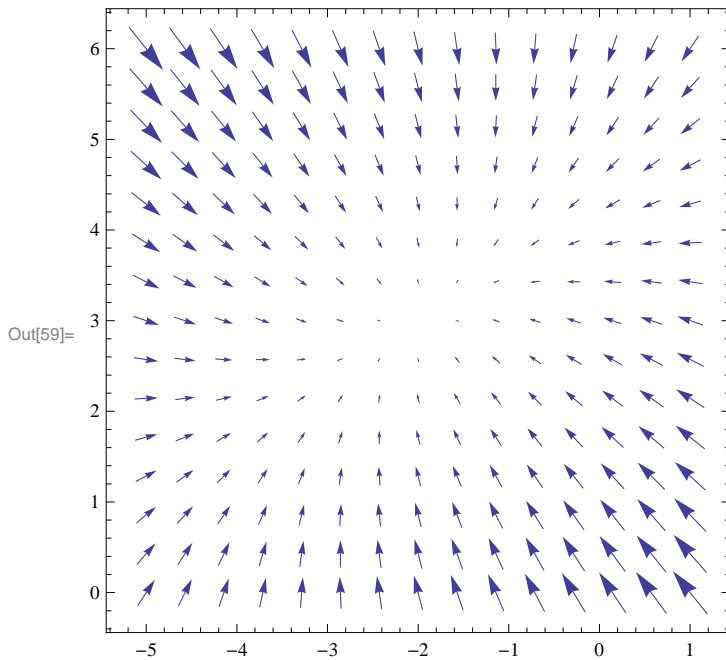
In Mathematica, we can isolate components of a vector by using the double bracket notation; h[[1]] means "use the first (or x) component of the vector h"; h[[2]] takes the second (or in this case, the y) component of the vector. The "%" sign instructs Mathematica to use the immediately preceding output; that is, Mathematica is using the output of line [40] in the Solve statement. Therefore, the expression "%[[1]]==0" means take the first component of the immediately preceding output and set it equal to zero; when solved simultaneously with "%[[2]]==0" we are able to find the (x,y) coordinates of the maximum elevation.

### ***Plotting Vector Fields in Mathematica***

We can use Mathematica to plot vector fields. If you are using version 6.0 of Mathematica (as I did in earlier write ups), you will need to load the vector field plotting package using the command Needs["VectorFieldPlots`"]. If you are using Mathematica 7.0 (as I am in writing this document) then the vector field plotting routine is loaded in the basic Mathematica kernel (and requires no "Needs" call).

In Mathematica 7.0, we use the VectorPlot function. This function requires the components of the vector to be plotted along with the plot intervals. Since we want to plot the gradient vector, we input in list form (that means we use braces) We will plot over the same interval used in the contour plots above :

```
In[59]:= VectorPlot[{-6 x + 2 y - 18, 2 x - 8 y + 28}, {x, -5, 1}, {y, 0, 6}]
```

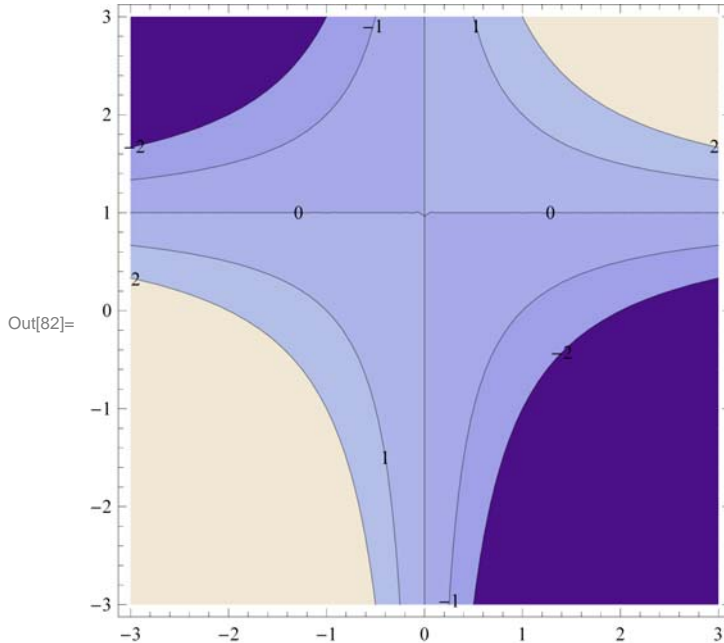


The arrows in the plot above show the magnitude and direction of the gradient at each  $(x, y)$  location shown. Notice how the arrows become shorter toward the center; this means the steepness of the gradient decreases as you travel from the edges of the graph to the center. Since the graph is oriented so that the top of the hill  $(-2, 3)$  is at the center of the plot, this corresponds to walking up the hill to its highest point. As you would expect, the steepness of the hill decreases as you reach the top, until the slope (or rate of change of elevation) becomes zero at the very top of the hill.

***Putting it all together, a worked example of Boas, p. 295/10:***

We are given a temperature field described by  $T = xy - x$  where  $x$  and  $y$  are the standard Cartesian coordinates. We are asked to show isothermals (contour lines) for  $T = 0, 1, 2, -1$  and  $-2$ , as well as finding the direction where temperature varies most rapidly at the point  $(1, 1)$  along with the magnitude of that change.

```
Clear[temp, x, y, z]
temp = x y - x;
ContourPlot[temp, {x, -3, 3}, {y, -3, 3}, ContourLabels -> All, Contours -> {-2, -1, 0, 1, 2}]
```



We are familiar with the use of "ContourLabels"; "Contours" allows us to specify the exact contours we wish to plot. We should not be surprised that the contour lines look like hyperbolae, since we can write the temperature field as :

$$T = x y - x = x (y - 1) \Rightarrow y - 1 = \frac{T}{x} \quad (14)$$

Since each contour line is for a set value of  $T$ , we are constructing a series of hyperbolae of the form  $y - 1 = T/x$ .

We obtain information about the rate of change of the temperature field by computing the gradient :

```
In[84]= Grad[temp, Cartesian[x, y, z]]
```

```
Out[84]= {-1 + y, x, 0}
```

So that the direction of greatest temp flow is  $(y - 1) \hat{x} + x \hat{y}$ . It is easy to see that at  $(1,1)$  this direction is simply  $+\hat{y}$ , and that the magnitude of  $\nabla T$  at  $(1,1)$  is 1.

We can also use this information to determine the direction of heat flow in the region. For instance, what will be the heat flow along a contour line? The answer is zero; since the temperature is the same all along a specific contour line, there is no  $dT/ds$  (where  $s$  is a coordinate of length along the contour line) and therefore no heat flow. Heat will flow in the direction of greatest temperature change, or, in the direction of the gradient of the field, and this will be perpendicular to the contour lines. Recall that the direction of the gradient at  $(1,1)$  is in the  $+\hat{y}$  direction, which is perpendicular to the  $T=0$  isotherm.