SOME NOTES ON THE HYPERBOLIC TRIG FUNCTIONS SINH AND COSH

Basic Definitions

In homework set #2 one of the questions involves basic understanding of the hyperbolic functions sinh and cosh. We will use this write - up to review those basics, and also to preview a bit of what we will learn when we study differential equations later in the course.

The hyperbolic functions are defined as :

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

We can find the derivatives of each function :

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \tag{2}$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \tag{3}$$

Compare these results with the well knows results of differentiating the basic trig functions :

$$\frac{d}{dx}(\sin x) = \cos x \tag{4}$$

$$\frac{d}{dx}(\cos x) = -\sin x \tag{5}$$

Therefore, it should be easy to show :

$$\int \cosh x \, dx = \sinh x \tag{6}$$

$$\int \sinh x \, dx = \cosh x \tag{7}$$

Mathematica Interlude

We verify all these results via *Mathematica*:

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In[88]:= D[{Cosh[x], Sinh[x]}, x]

Out[88]= {Sinh[x], Cosh[x]}

In[89]:= Integrate[{Cosh[x], Sinh[x]}, x]

Out[89]= {Sinh[x], Cosh[x]}

In[90]:= D[{Cos[x], Sin[x]}, x]

Out[90]= {-Sin[x], Cos[x]}

In[91]:= Integrate[{Cos[x], Sin[x]}, x]

Out[91]= {Sin[x], -Cos[x]}
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Note how we use brackets to perform several differentiations or integrations at once. The command "D[{Cos[x], Sin[x]}, x]" instructs Mathematica to differentiate with respect to x, the list of functions cos x and sin x. The output is in the form of a list; for example, input and output lines 90 tell you that the derivative of cos x is - sin x and the derivative of sin x is cos x.

The Pythagorean Theorem

We recall from basic trig that the Pythagorean theorem applied to sin and cos yields the well known :

$$\cos^2 x + \sin^2 x = 1$$

But for the hyperbolic functions, we have :

$$\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

so that we have :

$$\mathbf{\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1} \tag{8}$$

Equation (8) allows us to see the connection between these functions and hyperbolae; the defining equation of a hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(Compare this to the equation for a circle of radius a; $x^2 + y^2 = a^2$).

Graphs of the hyperbolic functions

Let' s use this as an opportunity to introduce basic graphing techniques in Mathematica. We will plot on one set of axes the curves for cosh x and sinh x. The graph will be over the interval - 3 < x < 3 :

In[92]:=

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Plot[{Cosh[x], Sinh[x]}, {x, -3, 3}]
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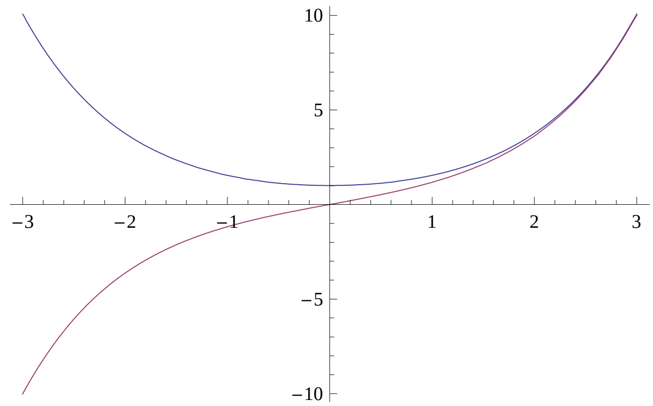
Out[92]=



Note that cosh x is an even function and that sinh x is an odd function. Look at the behavior of cosh and sinh as x grows large; can you explain why the curves approximate each other for x > 3?

Notice also that Mathematica will plot curves in different colors when instructed to graph more than one curve on a set of axes; the curve in blue will always correspond to the first function listed in the plot call; the red (purple?) curve will always correspond to the second color. (Of course, you can plot many more functions on a single set of axes.)

Applications to Differential Equations

Differential equations are used extensively in physics; understanding their properties and solutions is critical in any study of physics more advanced than first year courses. We can see easily from the properties of the hyperbolic and trig functions presented in equations (2) - (5) that :

$$\frac{d^2(\cos x)}{dx^2} = -\cos x \qquad \frac{d^2(\sin x)}{dx^2} = -\sin x$$

Therefore, sin and cos satisfy the simple differential equation :

$$\frac{d^2 y}{dx^2} + y = 0 \tag{9}$$

Equation (9) is a very important equation in physics; you might not recognize it in this form, but this is the essential form of Hooke's Law written as a differential equation. You likely encountered Hooke's Law as something like:

$$F = -kx$$

where F is the force of a spring acting on a mass (m), k is the spring constant, and x is the displacement of the mass from equilibrium. If we combine this with Newton' s second law, F = ma, we have :

$$F = kx = -ma \Rightarrow ma + kx = 0$$

Now, if we recast this in terms of calculus, we know that acceleration is the second derivative of of the position, so we have :

$$m\frac{d^2 x}{dt^2} + k\,x = 0 \;\text{ or }\; \frac{d^2 x}{dt^2} + \omega^2 x = 0 \;\text{ where } \omega = \sqrt{\frac{k}{m}}\,. \tag{10}$$

And apart from the fancy new constant we have introduced, eq. (10) is of the same form as eq. (9). Eq. (10) asks us to find a function, which when differentiated twice and added to the original function (times a constant), sums to zero. We already know that sin and cos will satisfy this, so our solution to the differential form of Hooke' s Law is

$$x(t) = A\cos(\omega\, t) + B\sin(\omega\, t) \tag{11}$$

where A and B are constants that are determined by the initial conditions of the particular system we are studying. The solution in eq. (11) is the very familiar harmonic oscillator; let' s assume for simplicity and specificity that A = B =1 and that ω=2. Then we can plot these solutions:

In[102]:=

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Clear[a, b, w, x]
a = 1; b = 1; w = 2;
Plot[ a Cos[w t] + b Sin[w t], {t, 0, 6 π}]
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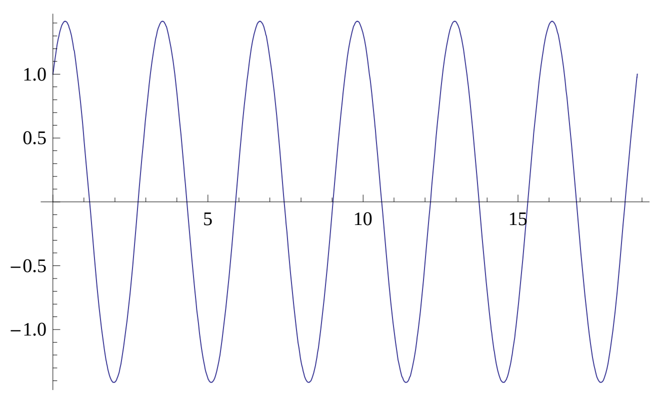
Out[104]=



As you would anticipate, the curve is that of a harmonic oscillator.

Now, what differential equation is satisfied by hyperbolic functions? Using the properties of their derivatives, it is easy to see that :

$$\frac{d^2 (\sinh x)}{dx^2} = \sinh x \text{ and } \frac{d^2 (\cosh x)}{dx^2} = \cosh x$$

We see from these relations that the second derivative of the hyperbolic function equals the original function, or stated in terms of a differential equation :

$$\frac{d^2 y}{dx^2} = y \Rightarrow \frac{d^2 y}{dx^2} - y = 0 \tag{13}$$

and the solutions to this equation are sinh x and cosh x. So we can write :

$$\frac{d^2 y}{dx^2} - y = 0 \Rightarrow y = A \sinh x + B \cosh x = A\left(\frac{e^x - e^{-x}}{2}\right) + B\left(\frac{e^x + e^{-x}}{2}\right) \tag{14}$$

where again A and B are constants. We can rewrite this by regrouping the exponential terms as :

$$y = c_1 e^x + c_2 e^{-x} \tag{15}$$

where c_1 and $c_2$are the new constants. You could for instance encounter eq. (13) if you had a situation where a particule was acted on by a force whose magnitude was proportional to the distance of the particle from an origin. (Imagine a particle free to move in a tube that is rotating with angular velocity ω; if the particle is a distance r from the origin, the force on the particle is proportional to ω^2r, and Newton's second law would yield equation (13).)

Probably all of you encountered sin, cos, sinh and cosh before you studied calculus. But imagine if you had never seen these functions before you studied differentiation; you would think of sin and cos as those interesting functions that satisfy y"+y=0. In the last month of the course we will study Legendre's equation (and possibly Bessel's equation); these are both important equations in physics (since they arise in the analysis of many important problems in physics) and are satisfied by special functions whose properties we will investigate in some depth.