REVIEW OF INTEGRATION

Trig Functions and Integration by Parts

Overview

In this note we will review how to evaluate the sorts of integrals we encounter in evaluating Fourier series. These will include integration of trig functions and also a quick review of integration by parts.

Derivatives of Trig Functions

Let' s begin with a quick review of the derivatives of the sin and cos functions, since knowing those will help us verify the results of our later integrations. We know from first year calculus :

$$\frac{d}{dx}(\sin(nx)) = n\cos(nx); \quad \frac{d}{dx}(\cos(nx)) = -n\sin(nx) \tag{1}$$

We can verify these statements using Mathematica :

```
D[{Sin[n x], Cos[n x]}, x]
{n Cos[n x], -n Sin[n x]}
```

Indefinite Integrals of Trig Functions

The results of the differentiations above suggest that the integrals of sin (nx) and cos (nx) are (ignoring any constants of integration) :

$$\int \sin(n\,x)\,dx = \frac{-1}{n}\cos(n\,x); \quad \int \cos(n\,x)\,dx = \frac{1}{n}\sin(n\,x) \tag{2}$$

We verify via Mathematica :

```
Integrate[{Sin[n x], Cos[n x]}, x]
```

$$\left\{-\frac{\mathrm{Cos}[n\,x]}{n}, \frac{\mathrm{Sin}[n\,x]}{n}\right\}$$

And we also verify by differentiating the results to show they equal the original integrand :

```
D[{-1 / n Cos[n x], 1 / n Sin[n x]}, x]
{Sin[n x], Cos[n x]}
```

As you have already seen, we encounter integrals involving sin (nx) and cos (nx) very frequently in our studies of Fourier series.

Integration by Parts

Integration by parts is a commonly used technique of integration that can be particularly useful if the integrand is a product of two functions. Suppose your integrand is of the form :

$$\int u\, dv \tag{3}$$

where both u and v are functions of the same variable, then the well known statement of integration by parts is :

$$\int u\, dv = uv - \int v\, du \tag{4}$$

Let' s work a few examples to show how we apply this technique. Consider the integral :

$$\int x\, e^x\, dx \tag{5}$$

The integrand is a product, so we should investigate whether integration by parts will be useful. Our first step is to decide which term in the integral in (5) corresponds to u and which corresponds to dv. Let' s work this example through . In (5), I will set u = x and dv = e^xdx. This allows me to determine expressions for du and v :

$$\begin{gathered} u = x \Rightarrow du = dx \\ dv = e^x\, dx \Rightarrow v = e^x \end{gathered} \tag{6}$$

We get the expression for v by integrating the expression for dv. Now that we can write u, v, du and dv, we can use eq. (4) to solve our integral :

$$\int \underbrace{x}_{u}\, \underbrace{e^x\, dx}_{dv} = \underbrace{x\, e^x}_{uv} - \int \underbrace{e^x}_{v}\, \underbrace{dx}_{du} \tag{7}$$

The final integral on the right is elementary, so our complete answer is :

$$\int x\, e^x\, dx = x\, e^x - \int e^x\, dx = xe^x - e^x = e^x\,(x-1) \tag{8}$$

Which we quickly verify both by direct Mathematica integration and also by differentiating the result :

```
Integrate[x Exp[x], x]
```

$e^x\,(-1+x)$

```
D[x Exp[x] - Exp[x], x]
```

$e^x\, x$

What would have happened if we initially chose u = e^x and dv=x dx, then we would have:

$$du = e^x\, dx;\; v = \frac{x^2}{2}$$

and our integration by parts would yield :

$$\int xe^x\, dx = \frac{x^2\, e^x}{2} - \frac{1}{2}\int x^2\, e^x\, dx \tag{9}$$

and our second integral would be more complicated than the first. We want our initial choice of u to produce the simplest possible integral on the right side of our integration by parts. Let' s try an example involving trig functions :

$$\int x \sin(n\,x)\,dx \tag{10}$$

We know that if we set u = x, then du = dx and we can imagine our second integral will be straightforward. On the other hand, if we set u = sin (nx) and dv = x dx, then our second integral will be very complicated. This tells us to assign :

$$\begin{gathered} u = x \;\Rightarrow\; du = dx \\ dv \;=\sin(n\,x)\,dx \;\Rightarrow v = -\frac{1}{n}\cos(n\,x) \end{gathered} \tag{11}$$

And we proceed via parts :

$$\begin{gathered} \int \underbrace{x}_{u}\,\underbrace{\sin(n\,x)\,dx}_{dv} \;=\; \underbrace{x}_{u}\,\underbrace{\left(\frac{-1}{n}\cos(n\,x)\right)}_{v} \;-\; \int \underbrace{\left(\frac{-1}{n}\cos(n\,x)\right)}_{v}\,\underbrace{dx}_{du} \;= \\ \frac{-x}{n}\cos(n\,x) \;+\; \frac{1}{n}\int \cos(n\,x)\,dx \;= \\ \frac{-x}{n}\cos(n\,x) + \frac{1}{n^2}\sin(n\,x) \end{gathered} \tag{12}$$

Verifying :

```
Integrate[x Sin[n x], x]
```

$$-\frac{x\,\mathrm{Cos}[n\,x]}{n} + \frac{\mathrm{Sin}[n\,x]}{n^2}$$

The integral of $\int x \cos(n\,x)\,dx$ is easily understood with a selection of u=x and dv = cos(n x) dx:

$$\begin{gathered} \int x \cos(n\,x)\,dx \;=\; x\left(\frac{1}{n}\sin(n\,x)\right) - \frac{1}{n}\int \sin(n\,x)\,dx = \\ \frac{x\sin(n\,x)}{n} + \frac{\cos(n\,x)}{n^2} \end{gathered} \tag{13}$$

Verifying :

```
Integrate[x Cos[n x], x]
```

$$\frac{\mathrm{Cos}[n\,x]}{n^2} + \frac{x\,\mathrm{Sin}[n\,x]}{n}$$

Let' s try a slightly more complicated case :

$$\int x^2\,e^x\,dx \tag{14}$$

We recognize this integrand also as a product, but we will see that we will have to employ integration by parts twice. Making the obvious substitions of u = x^2 and dv = e^x dx:

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx \tag{15}$$

And we realize the last integral in (14) will require integration by parts. Fortunately, we have already done this integral (see eq.(8) above), and we can write :

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx = x^2 e^x - 2 \left(x e^x - e^x\right) = e^x \left(x^2 - 2x + 2\right) \tag{16}$$

or :

```
Integrate[x^2 Exp[x], x]
```

$$e^x \left(2 - 2x + x^2\right)$$

Integration by Parts with Definite Integrals

For our work with Fourier series, we are of course interested in evaluating these integrals between limits. I show below how one would do the integral of eq. (10) with limits of 0 to π. Not surprisingly, we just evaluate each term at these limits and find:

$$\int_0^\pi x \sin(n x) \, dx = \left.\frac{-x \cos(n x)}{n}\right|_0^\pi + \left.\frac{\sin(n x)}{n^2}\right|_0^\pi =$$

$$\frac{-1}{n}[\pi \cos(n \pi) - 0] + \frac{1}{n^2}[\sin(n \pi) - \sin(0)] =$$

$$\frac{-\pi}{n} (-1)^n \tag{17}$$

Verifying :

```
Integrate[x Sin[n x], {x, 0, π}, Assumptions → Element[n, Integers]]
```

$$-\frac{(-1)^n \pi}{n}$$

One Final Example

Let' s see how we might approach a slightly more complex problem. Suppose we wish to integrate :

$$\int e^x \sin(n x) \, dx \tag{18}$$

Our integrand is a product, but both of the products are non - polynomials. Let' s see how we can attack this integral via parts. We set :

$$u = e^x \Rightarrow du = e^x \, dx$$

$$dv = \sin(n\,x)\,dx \;\Rightarrow\; v \;=\; \frac{-1}{n}\cos(nx)$$

Using these substitutions, our integral (which we define as I) becomes :

$$I = \int e^x \sin(n\,x)\,dx \;=\; \frac{-1}{n}\cos(n\,x)\,e^x - \left(\frac{-1}{n}\right)\int e^x \cos(n\,x)\,dx \tag{19}$$

And we are about to lose hope when we realize our integration by parts has produced another integral which also involves a product of transcendental terms. But, before we despair, let' s try to integrate the final integral in (19) by parts, using in this case the substitutions :

$$u = e^x \;\Rightarrow\; du = e^x\,dx;\;\; dv \;=\; \cos(n\,x)\,dx \;\Rightarrow\; v \;=\; \frac{1}{n}\sin(n\,x)$$

With this second substition, we have :

$$\begin{aligned} I = \int e^x \sin(n\,x)\,dx \;&=\; \frac{-1}{n}\cos(n\,x)\,e^x - \left(\frac{-1}{n}\right)\int e^x \cos(n\,x)\,dx = \\ &\frac{-1}{n}\cos(n\,x)\,e^x \;+\; \frac{1}{n}\left[\frac{1}{n}\sin(n\,x)\,e^x \;-\; \frac{1}{n}\int e^x \sin(nx)\,dx\right] = \\ &\frac{-1}{n}\cos(n\,x)\,e^x \;+\; \frac{1}{n^2}\sin(n\,x)\,e^x - \frac{1}{n^2}\int e^x \sin(n\,x)\,dx \end{aligned} \tag{20}$$

And now this is really depressing, because we have yet again produced an integral with a product of transcendental terms ... but ... look at this last integral, it is simply our original integral I with the coefficient of $1/n^2$, so we can rewrite (20) as:

$$I = \;\frac{-1}{n}\cos(n\,x)\,e^x \;+\; \frac{1}{n^2}\sin(n\,x)\,e^x - \frac{1}{n^2}\,I$$

where $I \;= \int e^x \sin(n\,x)\,dx$. Now, we just algebraically collect all terms in I:

$$\left(1 + \frac{1}{n^2}\right)I \;=\; e^x\left(\frac{\sin(n\,x)}{n^2} - \frac{\cos(n\,x)}{n}\right)$$

and after a little algebra we solve for I :

$$I = \;\int e^x \sin(n\,x)\,dx \;=\; \frac{e^x\,(\sin(n\,x) \;-n\cos(n\,x))}{n^2+1} \tag{21}$$

And we beseech Mathematica for a response :

```
Integrate[Exp[x] Sin[n x], x]
```

$$\frac{e^x\,(-n\,\mathrm{Cos}[n\,x] + \mathrm{Sin}[n\,x])}{1+n^2}$$

You may recall that we solved very similar integrals using techniques from complex numbers on the first homework set of the term.