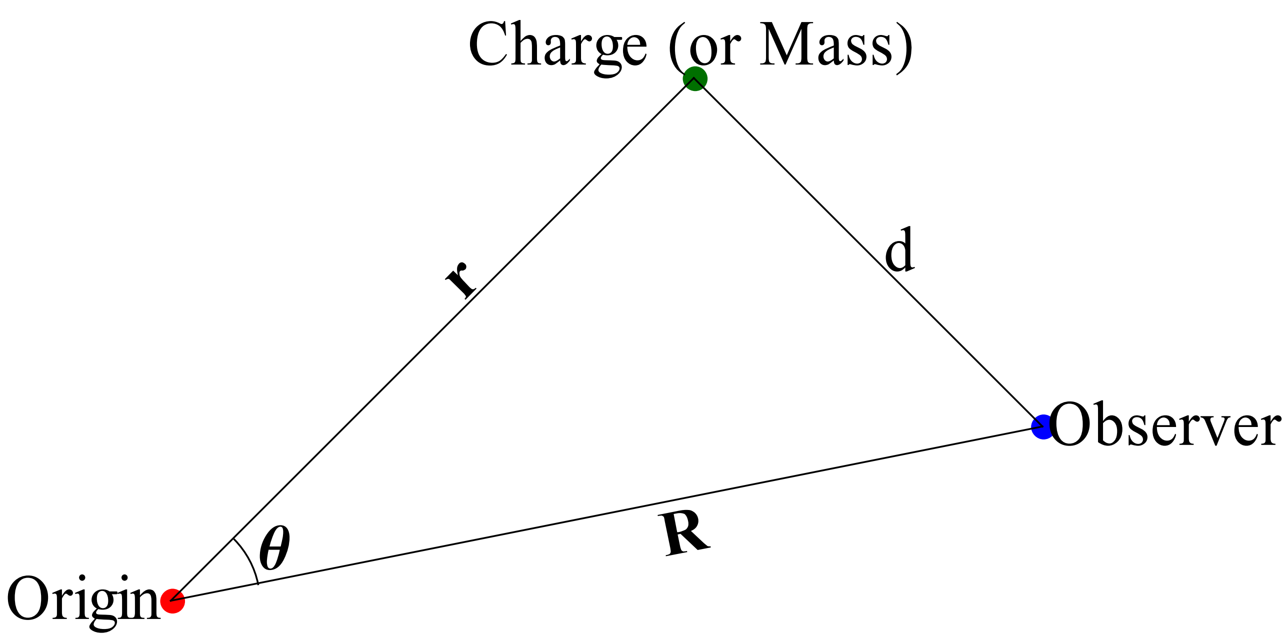
LEGENDRE POLYNOMIALS
GENERATING FUNCTIONS
AND MULTIPOLE EXPANSIONS

In class we have already seen how Legendre polynomials represent a solution to the differential equation:

$$(1 - x^2)y'' - 2xy' + l(l+1)y = 0 \qquad (1)$$

We show how a completely different analysis, one arising from a well known physical situation, also yields the derivation of Legendre polynomials. This analysis will also allow us to write the **generating function** for Legendre polynomials. Understanding the generating function will allow us to investigate these polynomials more deeply, and allow us to find many useful relationships.

First, we start with a well known physical situation as depicted below:



This figure shows a particle at the head of vector **r** with respect to a fixed origin. An observer is located at the end of the vector denoted as **R.** Quite often in physics, we want to describe the potential field measured at **R** generated by the particle located at **r**. In this case it does not matter if we are interested in the potential arising from an electric or gravitational field of the particle; since both fields follow inverse square laws, their potentials are expressible in the same mathematical format.

We know from basic physics that the potential of a $1/r^2$ field goes as $1/r$, so that the potential of the particle at the observer will go as K/d, where K is some constant and d is the magnitude of the vector **d** as shown in Fig. 1.

If we wish to express the potential at the observer in terms of the coordinate system in which the origin is at (0,0,0), we need to write **d** in terms of **r**, **R**, and θ. The law of cosines tells us that:

$$d^2 = r^2 + R^2 - 2rR\cos\theta \Rightarrow d = \sqrt{r^2 + R^2 - 2rR\cos\theta} \qquad (2)$$

We assume $R > r$ and factor out R^2 from (2) to obtain:

$$d = R\sqrt{1 + (r/R)^2 - 2(r/R)\cos\theta} \qquad (3)$$

Following the notation used in the text, set $h=r/R$ and $x=\cos\theta$ to produce:

$$d = R\sqrt{1 - 2hx + h^2} \qquad (4)$$

and we repeat the admonition of the text to note that x is not related to the Cartesian coordinate, but is merely a term used to represent cos θ.

With our expression in (4), it is clear that we can write the potential measured by the observer as:

$$V = K/d = \frac{K}{R\sqrt{1 - 2hx + h^2}} \qquad (5)$$

As befits a study of series solutions of differential equations, we next take the power series expansion of (5). First, let's review a few results you should be familiar with. The radical in (5) can be thought of as $\frac{1}{\sqrt{1-y}}$ where y is $(2hx-h^2)$. The power expansion of $\frac{1}{\sqrt{1-y}}$ is given by:

$$\frac{1}{\sqrt{1-y}} = f(0) + f'(0)y + \frac{f''(0)}{2!}y^2 + \frac{f'''(0)}{3!}y^3 + \ldots \qquad (6)$$

Taking the appropriate derivatives of $\frac{1}{\sqrt{1-y}}$ and evaluating at $y = 0$ yields:

$$f(y) = \frac{1}{\sqrt{1-y}} = 1 + \frac{y}{2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{y^2}{2!} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{y^3}{3!} + \ldots$$
$$= 1 + \frac{y}{2} + \frac{3y^2}{8} + \frac{5y^3}{16} + \frac{35y^4}{128} + \ldots \quad (7)$$

If we set $y = 2hx - h^2$ and substitute into (7), and expand each term we get:

$$\frac{1}{\sqrt{1-y}} =$$

$$\frac{1}{\sqrt{1-2hx+h^2}} = 1 + \frac{(2hx-h^2)}{2} + \frac{3(2hx-h^2)^2}{8} + \frac{5(2hx-h^2)^3}{16} + \frac{35(2hx-h^2)^4}{128} + \ldots$$

$$= 1 + hx - \frac{h^2}{2} + \frac{3h^2x^2}{2} + -\frac{3h^3x}{2} + \frac{3h^4}{8} + \frac{5h^3x^3}{2} - \frac{15h^4x^2}{4} + \frac{15h^5x}{8} - \frac{5h^6}{16} + O(h^4) \quad (8)$$

If we group the terms in (8) according to powers of h, we get:

$$\frac{1}{\sqrt{1-y}} = \frac{1}{\sqrt{1-2hx+h^2}} = 1 + hx + h^2(\frac{3x^2}{2} - \frac{1}{2}) + h^3(\frac{5x^3}{2} - \frac{3x}{2}) + \ldots \quad (9)$$

Eq. (9) is presented as a power series in powers of h. We are now very familiar with different types of power series, and know the next question is to investigate the nature of the coefficients of h. If we look at these coefficients carefully, we see that:

$$a_0 = 1$$
$$a_1 = x$$
$$a_2 = \frac{1}{2}(3x^2 - 1)$$
$$a_3 = \frac{1}{2}(5x^3 - 3x)$$
$$a_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

(this is the expression we would obtain for a_4 if we expanded and evaluated the $\frac{35(2hx-h^2)^4}{128}$ term from eq. (8) above.

We recognize each coefficient a_n as the Legendre Polynomial of order n, so that we can write our series expansion (9) as:

$$\frac{1}{\sqrt{1-2hx+h^2}} = P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \ldots = \sum_{l=0}^{\infty} P_l(x)h^l \quad (10)$$

This is a very important relationship, because it allows us to write Legendre functions in terms of the power expansion of a well known function. We define:

$$g(x,h) = \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{l=0}^{\infty} P_l(x)h^l \quad \text{for values of } |h| < 1 \quad (11)$$

and we call g(*x, h*) the ***generating function for Legendre polynomials*** because we can generate Legendre Polynomials by expanding g(*x, h*) as a power series.

So, we can look back at our expression for potential observed at the location **R** in terms of Legendre polynomials, and write (5) as:

$$V = \frac{K}{d} = \frac{K}{R\sqrt{1-2hx+h^2}} = \frac{K}{R}\sum_{l=0}^{\infty} P_l(x)h^l = K\sum_{l=0}^{\infty} \frac{r^l P_l(x)}{R^{l+1}} \qquad (12)$$

where we have made use of the substitution $h=r/R$ in the last step in (12).

Using the generating function

Parity of Legendre Polynomials

We can use the generating function to investigate the behavior of Legendre polynomials. For instance, suppose we consider $g(-x, -h)$, we use the expression in (11) to show:

$$g(-x,-h) = \frac{1}{\sqrt{1-2(-x)(-h)+(-h)^2}} = \frac{1}{\sqrt{1-2xh+h^2}} \Rightarrow g(-x,-h) = g(x,h) \quad (13)$$

The identity in (13) allows us to write eq. (12) in the form:

$$\sum_{l=0}^{\infty} P_l(x)h^l = \sum_{l=0}^{\infty} P_l(-x)(-h)^l \quad (14)$$

This allows us to write immediately:

$$P_l(-x) = (-1)^l P_l(x) \quad (15)$$

Eq. (15) shows us an important property of Legendre polynomials; namely, they are even functions for even values of l and odd functions for l = odd.

Recurrence Relations

The generating function allows us to establish relationships among Legendre polynomials; these relations are shown on p. 570 in Boas (eqs. 5.8a-f). Their derivation in each case involves manipulating the generating function in some way.

We will begin by taking our definition of the generating function:

$$g(x,h) = \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{l=0}^{\infty} P_l(x)h^l$$

and differentiate both sides with respect to *h*:

$$\frac{\partial g}{\partial h} = \frac{-1(-2x+2h)}{2(1-2hx+h^2)^{3/2}} = \sum_{l=1}^{\infty} lP_l(x)h^{l-1} \qquad (16)$$

Notice that the summation on the right is now from l=1. We can rewrite (16) as:

$$\frac{(x-h)}{\sqrt{1-2hx+h^2}} = (1-2hx+h^2)\sum_{l=1}^{\infty} lP_l(x)h^{l-1} \qquad (17)$$

Remembering that $g(x,h) = \dfrac{1}{\sqrt{1-2hx+h^2}}$, the left hand side of (17) becomes:

$$(x-h)\sum_{l=0}^{\infty} P_l(x)h^l = (1-2hx+h^2)\sum_{l=1}^{\infty} lP_l(x)h^{l-1} \qquad (18)$$

We can multiply both sides of (18) term by term, and produce the following summations; note carefully the lower limits. From this point, I will assume an upper limit of infinity for each summation (again, this is done for ease of typing):

$$x\sum_{l=0} P_l(x)h^l - \sum_{l=0} P_l(x)h^{l+1} = \sum_{l=1} lP_l(x)h^{l-1} - 2x\sum_{l=1} lP_l(x)h^l + \sum_{l=1} lP_l(x)h^{l+1} \quad (19)$$

We have done enough work with series solutions to know our next step is to equate terms with equal powers of *h*. We need to be careful to choose the proper value of *l* in each summation so that each summation produces the same exponent for *h*. For example, suppose we want to equate all h^3 terms; to produce h^3 terms in each summation, we need to set l=3 in the first sum; l=2 in the second, l=4 in the third sum (the first sum on the RHS); l=3 in the fourth sum, and l = 2 in the final sum. Using these values of *l* in (19), we get:

$$h^3[xP_3(x) - P_2(x)] = h^3[4P_4(x) - 2x(3)P_3(x) + 2P_2(x)] \qquad (20)$$

We can use eq. (20) to derive our final recursion relation. First, we immediately recognize we can divide each side by h^3. Second, we realize that (2) tells us the value of $P_4(x)$ in terms of $P_3(x)$ and $P_2(x)$. We can generalize (20) to any values of l, l-1 and l-2, and rewrite (20) as:

$$lP_l(x) = xP_{l-1}(x) + 2(l-1)xP_{l-1}(x) - P_{l-2}(x) - (l-2)P_{l-2} \qquad (21)$$

In eq. (21), the $lP_l(x)$ term corresponds to the $4P_4(x)$ term in (20); the $xP_{l-1}(x)$ term corresponds to the $xP_3(x)$ term in (20) and so on.

We can group these terms and produce finally our recursion relation:

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x) \quad (22) \quad \text{(5.8a on p. 570)}$$

This recursion relation is a very efficient way to calculate higher order Legendre polynomials if you know the two preceding polynomials. As an example, let's find $P_2(x)$ given that $P_1(x) = x$ and $P_0(x) = 1$. In solving for $P_2(x)$, we set l=2 and find:

$$2P_2(x) = 3xP_1(x) - (1)P_0(x) = 3x^2 - (1)(1) \Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad (23)$$

which reproduces the well known result for $P_2(x)$.

Let's try to derive one more recursion relation to show the utility of the generating function. We will derive the recursion relation shown in eq. (5.8b) on p. 570 in Boas. First, we start with the generating function (11), and differentiate it with respect to x and separately with respect to h. These differentiations yield:

$$\frac{\partial g}{\partial x} = \frac{-1(-2h)}{2(1-2hx+h^2)^{3/2}} = \frac{h}{(1-2hx+h^2)^{3/2}} \; ; \; \frac{\partial g}{\partial h} = \frac{-1(-2x+2h)}{2(1-2hx+h^2)^{3/2}} = \frac{(x-h)}{(1-2hx+h^2)^{3/2}}$$

(24)

We notice that both differentiations have the same denominator, so we can write:

$$\frac{1}{h}\frac{\partial g}{\partial x} = (1-2hx+h^2)^{-3/2} = \frac{1}{x-h}\frac{\partial g}{\partial h} \Rightarrow (x-h)\frac{\partial g}{\partial x} = h\frac{\partial g}{\partial h} \qquad (25)$$

Eq. (25) involves derivatives of the generating function *g(x,h)*; we can now differentiate the power series representation of *g(x, h)* and substitute those values into (25):

$$g(x,h) = \sum_{l=0} P_l(x)h^l \Rightarrow \frac{\partial g}{\partial x} = \sum_{l=0} P_l'(x)h^l \; ; \; \frac{\partial g}{\partial h} = \sum_{l=1} lP_l(x)h^{l-1} \qquad (26)$$

Substituting these relations into (25) we obtain:

$$(x-h)\sum_{l=0} P_l'(x)h^l = h\sum_{l=1} lP_l(x)h^{l-1} \Rightarrow x\sum_{l=0} P_l'(x)h^l - \sum_{l=0} P_l'(x)h^{l+1} = \sum_{l=1} lP_l(x)h^l \quad (27)$$

We know the routine by now; equate terms with equal powers of *h* on both sides of the equation, being careful to know which value of *l* to use in each summation to ensure equality of exponents of *h*. When we do this, the final identity in (27) becomes:

$$xP_l'(x) - P_{l-1}'(x) = lP_l(x) \quad (28)$$

which agrees with (5.8b) on p. 570 of Boas.

Derivation of further recursion relations involve this sort of general procedure; manipulating the generating function and combining with previous results to determine further relations. Notice that we can derive these recursion relations knowing only the form of the generating function (11); we do not need to know anything about the specific nature of the Legendre function to derive these relations. We will see that we can derive recursion relations for any special function as long as we know its generating function and are clever in manipulating it.

Legendre polynomials are the first type of ***special functions*** that we have studied in this course. Mathematical physics is replete with these functions which are solutions to differential equations that occur frequently in physics. ***Bessel functions*** often arise when there is cylindrical symmetry; ***Hermite functions*** are encountered in the study of the harmonic oscillator in quantum mechanics; ***Laguerre functions*** are the solutions to the description of an electron in the hydrogen atom in quantum mechanics. Once you are comfortable with one or two special functions, you will be able to work with any of these; they are all solutions to differential equations, all have generating functions, and the generating functions are useful in deriving important recursion relations for each special function.

The Electric Dipole

Let's consider now the electric dipole depicted in Fig. 2 below:

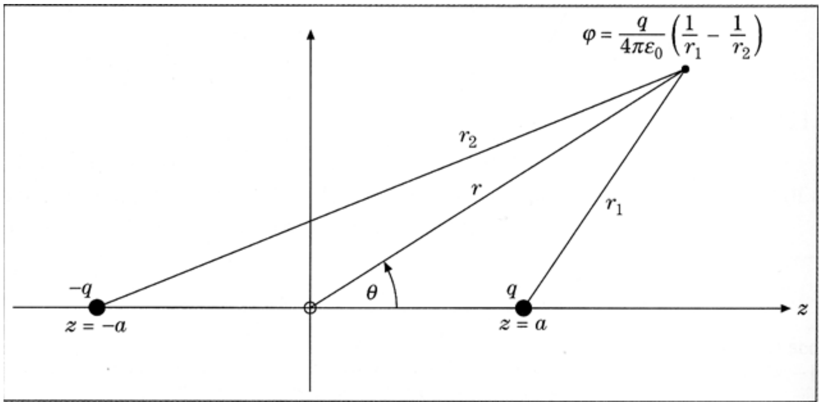


Fig. 2

Fig. 2 shows a classic dipole. A charge of $+q$ is found at (a, 0) along the x axis, and a charge of $-q$ is found at (-a,0) along the $-x$ axis. The $+q$ charge is a distance r_1 from an observer, and the charge $-q$ lies a distance r_2 from the same observer. The distance from the origin to the observer is r. How can we express the potential measured at the observer?

The electric potential due to a point charge is given simply as:

$$\phi = \frac{q}{4\pi\varepsilon_0 d} \quad (29)$$

where φ is the electric potential, q is the charge, ε_0 is the permittivity of free space, and d is the distance between the charge and the point of measurement. For the case depicted in Fig. 2, the potential is described by the equation at the top of the figure. The minus sign arises because of the charge of q at $(0,-a)$.

As we did before, we can write the distances r_1 and r_2 in terms of r and θ. Using the law of cosines, we can write:

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta \Rightarrow r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta} = r\sqrt{1 + (a/r)^2 - 2(a/r)\cos\theta}$$

$$(30)$$

$$r_2^2 = r^2 + a^2 + 2ar\cos\theta \Rightarrow r_2 = \sqrt{r^2 + a^2 + 2ar\cos\theta} = r\sqrt{1 + (a/r)^2 + 2(a/r)\cos\theta}$$

Note carefully the sign of the $2ar\cos\theta$ in r_2; this sign occurs because the angle opposite r_2 is 180-θ, and cos(180-θ) = -cos θ.

By substituting the relations in (30) into the expression for the dipole potential, we obtain:

$$\phi = \frac{q}{4\pi\varepsilon_0 r}\left(\frac{1}{\sqrt{1-2(a/r)\cos\theta+(a/r)^2}} - \frac{1}{\sqrt{1+2(a/r)\cos\theta+(a/r)^2}}\right) \qquad (31)$$

The expressions in parentheses should look very familiar to us; they are the generating functions for Legendre polynomials that we have seen earlier (eq. 11) Here, (*a/r*) takes the place of *h*, and cos θ replaces *x*. Referring again to eq. (11), we know we can express (31) in terms of Legendre polynomials:

$$\phi = \frac{q}{4\pi\varepsilon_0 r}\left(\sum_{l=0}^{\infty} P_l(\cos\theta)(a/r)^l - \sum_{l=0}^{\infty}(-1)^l P_l(\cos\theta)(a/r)^l\right) \qquad (32)$$

The two summations in (32) appear very similar; they differ only in the sign of *a*; (*a* is positive in the first summation; negative in the second). This difference in the sign of *a* is responsible for the $(-1)^l$ term in the second summation.

If we study (32) we can see that the term in parentheses goes to zero if *l* is even, so we have:

$$\phi = \frac{2q}{4\pi\varepsilon_0 r}\left[P_1(\cos\theta)(a/r) + P_3(\cos\theta)(a/r)^3 + P_5(\cos\theta)(a/r)^5 + \ldots\right] \qquad (33)$$

We can easily expand the result in (33) in terms of cos θ and powers of (*a/r)*. First, let's look at (33) and realize that if $r >> a$, the dominant term in the expansion will be the first term, so to first order, we can express the electric dipole in (33) as:

$$\phi = \frac{2q}{4\pi\varepsilon_0 r}\left[\cos\theta\,(a/r)\right] = \frac{2aq}{4\pi\varepsilon_0 r^2}\cos\theta \qquad (34)$$

Remember that $P_1(x) = x$, so $P_1(\cos\theta) = \cos\theta$. In (34), the term $2aq$ is the electric dipole moment.

A little bit of electrodynamics…

We can combine some of what we have learned in vector calculus with what we are learning now to address some problems in electrodynamics.

In (34) above we derive, to first order, the potential due to an electric dipole a distance *r* far from the origin. (Far from the origin, the first term in the expansion dominates.) What is the electric field due to the dipole at *r*?

First, we recall that conservative forces are derivable from the gradient of a scalar function, and since **E** is a conservative, field, we should not be surprised that:

$$\mathbf{E} = -\nabla\phi$$

Remembering that φ is a function of *r* and *θ*, we will compute both the *r* and *θ* components of the electric field. Recall also that the definition of the gradient in the plane polar coordinate system is:

$$\nabla\phi(q_1, q_2) = \sum_i \frac{1}{h_i}\frac{\partial\phi}{\partial q_i}\mathbf{q}_i = \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad (35)$$

where the q_i's are generalized coordinates, $\mathbf{q}_i$ represent unit vectors, and h_i are the scale factors.

Differentiating (34) according to (35), we get the following components of the **E** field at *r:*

$$E_r = -\frac{\partial}{\partial r}\left(\frac{2aq\cos\theta}{4\pi\varepsilon_0 r^2}\right) = \frac{4aq\cos\theta}{4\pi\varepsilon_0 r^3};\ and\ E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{2aq\cos\theta}{4\pi\varepsilon_0 r^2}\right) = \frac{2aq\sin\theta}{4\pi\varepsilon_0 r^3} \qquad (36)$$