COMPUTING FOURIER SERIES

Overview

We have seen in previous notes how we can use the fact that sin and cos represent complete orthogonal functions over the interval $[-\pi,\pi]$ to allow us to determine the coefficients of a Fourier series. Now, let's use this information to evaluate some examples of Fourier series. We will, of course, show how using *Mathematica* can make our lives easier, especially in the evaluation of integrals.

Our First Example

Let's begin by computing the Fourier series of $f(x) = x$ over the interval $[-\pi,\pi]$. Let's think about this for a moment; we are about to represent a straight line by an infinte sum of sinusoidal curves. We will first do the calculations outlined in previous notes, and then show that these sums in fact give us the line $y=x$ over the interval $[-\pi,\pi]$.

First, let's write down the general form of a Fourier series. We can write any function that is 2π periodic as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \qquad (1)$$

where we have to evaluate the coefficients :

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \qquad (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \qquad (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \qquad (4)$$

Evaluating these three coefficients will require that you be review integrating trig functions, and also review the value of trig functions at certain values (e.g., knowing the value of sin and cos at $0, \pm n\pi; \pm(n\pi/2)$).

Let's proceed with our particular example. Eqs. (2)-(4) will provide us with the coefficients that we substitute into (1) to find the Fourier series for $f(x)=x$. We will now compute these coefficients, and then produce the final Fourier series.

- **Finding the value of a_0 :**

We use eq. (2) to find the value of a_0 that we will use in our final Fourier series:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = \frac{1}{\pi} \left[\left(\frac{x^2}{2} \right)_{\pi} - \left(\frac{x^2}{2} \right)_{-\pi} \right] = 0 \tag{5}$$

Or, using Mathematica to verify :

```
(1 / π) Integrate[x, {x, -π, π}]
```

0

Some of you may have been able to solve this using symmetry considerations; more on the use of symmetry in another classnote.

■ **Finding the values of a_n :**

We use eq. (3) along with *Mathematica* to find the values of a_n :

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n\,x) \, dx \Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(n\,x) \, dx \tag{6}$$

```
Integrate[x Cos[n x], {x, -π, π}, Assumptions → Element[n, Integers]]
```

0

And we find that all over the coefficients a_n are zero; again, those of you proficient with the use of symmetry arguments might have seen this immediately. This result means that the only terms that will contribute to the Fourier series of f(x)=x will consist of sin (n x) terms. We now determine those coefficients.

■ **Finding the values of b_n:**

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n\,x) \, dx \Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(n\,x) \, dx \tag{7}$$

NOTE TO CLASS : The calculation that appears below is incorrect. Immediately below I will correct this error and proceed through the rest of the computation using the correct values.

```
Clear[n]
(1 / π) Integrate[x Sin[n x], {x, 0, π}, Assumptions → Element[n, Integers]]
```

$$-\frac{(-1)^n}{n}$$

Can you find the error in the Mathematica statement above? Look at the limits : I have integrated between 0 and π instead of between $-\pi$ and π. The correct calculation should be:

```
In[1]:= (1 / π) Integrate[x Sin[n x], {x, -π, π}, Assumptions → Element[n, Integers]]
```

$$\text{Out[1]=} \quad -\frac{2 (-1)^n}{n}$$

The correct answer is double the error above; exactly what you would expect from symmetry arguments. (See the classnote on

"Integrating Odd and Even Functions"). The correct version of the integral tells us that our coefficients are :

$$b_n = \frac{-2(-1)^n}{n} \tag{8}$$

We will see many expressions like this when we calculate Fourier coefficients. Let's write out the first few coefficients explicitly to see what they look like :

$$b_1 = -(-1)^1 = 2; \ b_2 = \frac{-(-1)^2 \, 2}{2} = -1; \ b_3 = \frac{-2(-1)^3}{3} = \frac{2}{3} \tag{9}$$

If we substitute these coefficients into eq. (1), we can write out explicitly the first three terms of the Fourier series for $f(x) = x$:

$$\begin{aligned} x &= b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots \\ &= 2\left[\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \ldots\right] \end{aligned} \tag{10}$$

Equation (10) represents the first three terms of the Fourier series for $f(x) = x$ over the interval $[-\pi,\pi]$. You might wonder how it is possible that a straight line can be represented by a sum of sine curves. Let's plot these three terms out and see if we can recognize the straight line y=x. Remember, these are only the first three terms of an infinite series, so we should not expect a perfect correlation with only three terms...so, here we go:

In[2]:= `Plot[2 (Sin[x] - Sin[2 x] / 2 + Sin[3 x] / 3), {x, -π, π}]`

Out[2]=
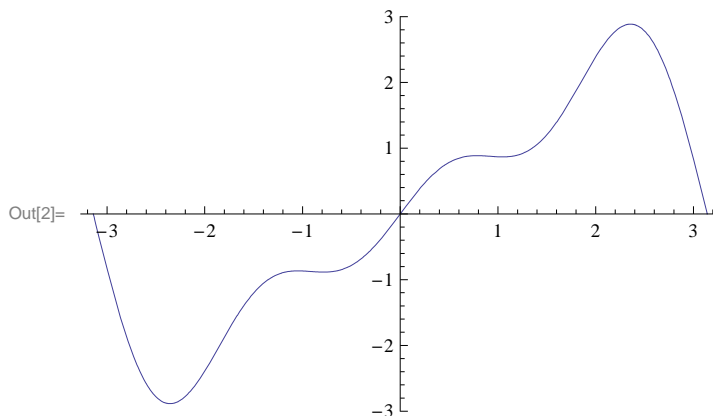
Well, sort of. Let's try a few more terms and see what we get :

In[3]:= `Plot[2 (Sin[x] - Sin[2 x] / 2 + Sin[3 x] / 3 - Sin[4 x] / 4 + Sin[5 x] / 5 - Sin[6 x] / 6 - Sin[7 x] / 7),`
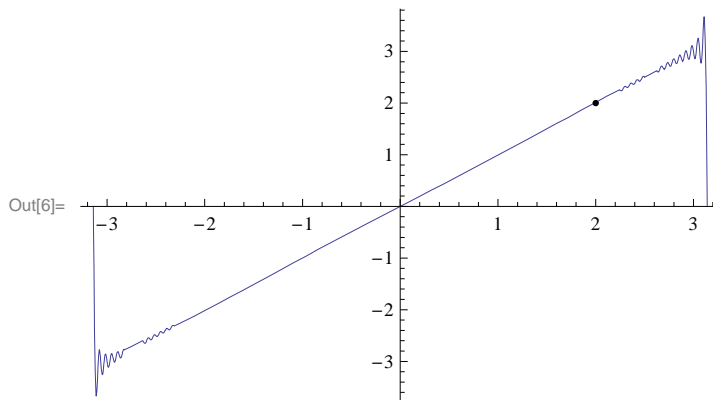   `{x, -π, π}]`

Out[3]=
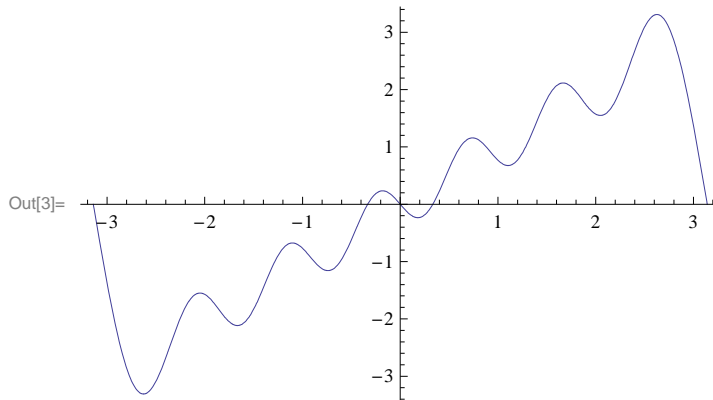
Closer. We could add more terms like this, but it would be laborious. Let' s learn how to write this series in closed form (e.g., equation (1)) and work with it. In closed form, this series becomes :

$$x = 2 \sum_{n=1}^{\infty} \frac{-(-1)^n \sin(n\,x)}{n} \qquad (11)$$

and we can now plot it using the Mathematica command for summation :

In[6]:= `Plot[Sum[-2 (-1) ^n Sin[n x] / n, {n, 1, 100}], {x, -π, π}, Epilog → Point[{2, 2}]]`

Out[6]=

Now that' s a straight line. (Quick *Mathematica* note: the final part of the Plot statement in line[6] introduces us to the "Epilog" option. This allows us to add something to a graph after it has been plotted. In this case, I wanted to plot the point {2,2} to show that it does in fact lie on the line y=x as expected. This helps verify that not only does our Fourier series produce a straight line, but the proper straight line.)

Let' s study input line[6] : it contains nested function calls. The "inner" call computes the sum indicated in eq. (11). The expression in braces "{n,1,100}" means to sum the function from n = 1 to 100. The outer call plots those calculations over the interval [-π,π].

The new (and current) version of *Mathematica* has a very powerful new feature, called "Manipulate". You should copy the following command exactly in your notebooks to see how it works. In this case, we will treat the upper index of the summation as a variable, in other words, we want to see how this Fourier series converges to f(x)=x as we vary the number of terms we use in the summation:

In[5]:= `Manipulate[Plot[Sum[-2 (-1)^n Sin[n x] / n, {n, 1, ul}], {x, -π, π}], {ul, 1, 200}]`

Out[5]=
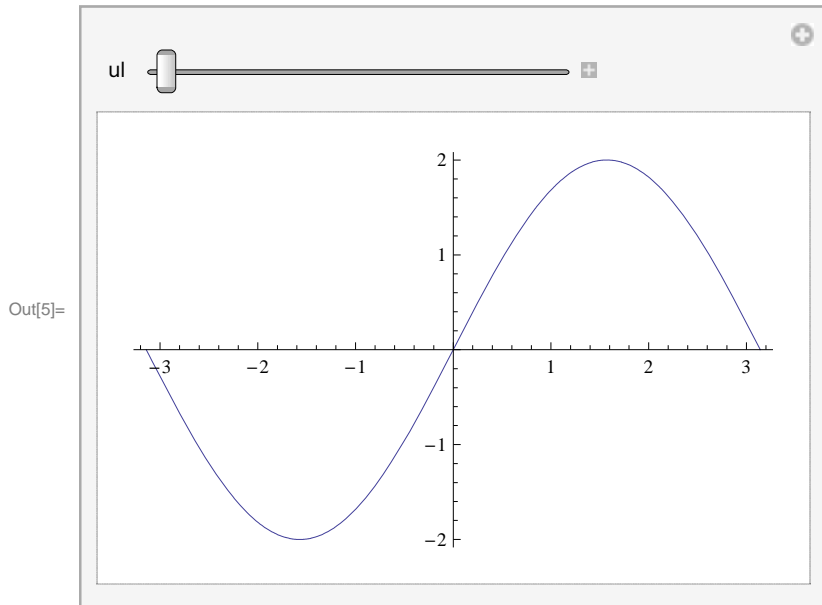
Notice that in input line[5] I used "ul" as a variable to indicate the upper limit of the sum; the final braces "{ul,1,200}" will cause Mathematica to recompute the plot. When ul = 1, the plot will be based on only 1 term in the summation; when ul = 20, the plot is based on summing the first 20 terms of the series. This is a powerful new function that allows for more detailed studies of these sorts of series than I have ever encountered. This is one reason why I wanted to use Mathematica this term.

A second Example

For this example, let' s work out the answer to question #7 on p. 355 of Boas. This is a little different from the first example since we have two regimes for our function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Because of this, we will need to be careful in evaluating our coefficients eqs. (2) - (4). It should be obvious that the contributions to all coefficients will be zero from the interval [-π,0], so we need to integrate only from 0 to π:

- **Finding a_0**

$$a_0 = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{\pi} \cdot \frac{\pi^2}{2} = \frac{\pi}{2} \Rightarrow \frac{a_0}{2} = \frac{\pi}{4} \tag{12}$$

- **Finding a_n:**

$$a_n = \frac{1}{\pi} \int_0^\pi x \cos(n x) \, dx \tag{13}$$

```
Clear[n]
(1 / π) Integrate[x Cos[n x], {x, 0, π}, Assumptions → Element[n, Integers]]
```

$$\frac{-1 + (-1)^n}{n^2 \pi}$$

The output line above tells us the values of all the coefficients a_n. But this expression bears further scrutiny. Notice that there is a term involving $(-1)^n$. This term will be -1 if n is odd, and +1 if n is even. Thus, the value of a_n will be zero for all even values of n, and will equal $-2/(\pi n^2)$ for all odd values of n. We will need to be very careful to take all this into account in writing out our Fourier series.

- **Finding b_n :**

$$b_n = \frac{1}{\pi} \int_0^\pi x \sin(n x)\, dx \tag{14}$$

```
Clear[n]
(1 / π) Integrate[x Sin[n x], {x, 0, π}, Assumptions → Element[n, Integers]]
```

$$-\frac{(-1)^n}{n}$$

- **Putting it together :**

We take our values for the coefficients calculated above and substitute into equation (1) to get our Fourier series. Writing out the first three (non - zero) terms in each part of the series, we get :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos(x)}{1^2} + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2}\right] + \left[\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + ...\right] \tag{15}$$

Let' s try to plot this sum for say the first 101 terms. We will have to be careful to take into consideration the fact that the cosine part of the series involves only odd integers. The simplest Mathematica way to handle this is :

```
Plot[(π / 4) - (2 / π) Sum[Cos[n x] / n^2, {n, 1, 101, 2}] +
   Sum[-(-1)^n Sin[n x] / n, {n, 1, 101}], {x, -π, π}]
```
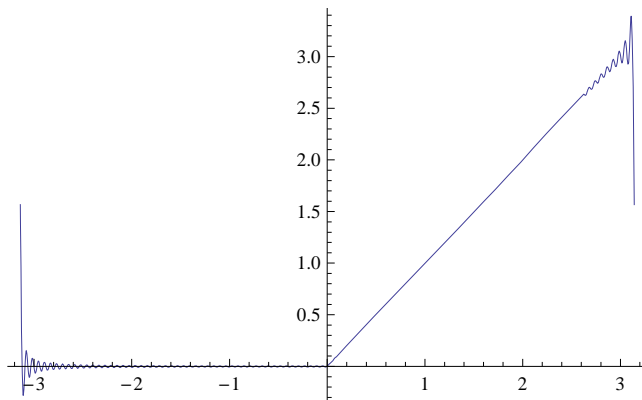

Consider the Sum bracket above : the notation "{n,1,101,2}" means to sum over n, from 1 to 101 indexing n by two. This is

equivalent to the statement :

$$\sum_{n=1,3,5\ldots}^{101} \frac{\cos(n\,x)}{n^2} \tag{16}$$