ORTHOGONALITY AND FOURIER SERIES-- PART II

Overview

In our last episode, we had gotten as far as recognizing the general form of a Fourier series :

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + \ldots \\ &\quad + b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \ldots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \end{aligned} \tag{1}$$

All Fourier series are expressible in this form. We need to learn how to find the coefficients a_o, a_n and b_n for each function f(x) that we wish to represent as a Fourier series. We will make extensive use of the orthogonal properties of sin and cos to determine these coefficients. Let's begin by determing the value of a_0. We can see in equation (1) that there is a relationship between a_0 and f(x), but how can we determine the value of only $a_0$without knowing the values of all the other coefficients a_n and b_n? The "trick" we will employ is one you will use many, many times as you work with Fourier and other types of series. Let's integrate each term in eq. (1) between the limits {$-\pi$, π} and divide the result by 2π. In other words:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx =$$

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} a_1 \cos(x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} a_2 \cos(2x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} a_3 \cos(3x)\,dx + \ldots \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} b_1 \sin(x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} b_2 \sin(2x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} b_3 \sin(3x)\,dx + \ldots \end{aligned} \tag{2}$$

Perfect. We have now created an infinite series of integrals of trig functions. How does this help?

Let' s look at each of these integrals carefully and remember that our goal is to figure out a simple way to evaluate the various coefficients in our Fourier series expansion. Once we have those coefficients, we can construct our completed series by substituting those coefficients back into eq. (1). Since a_0 is just a constant, the first integral on the right side of (2) is easily calculated:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \frac{1}{2\pi}\left(\frac{a_0}{2}\right)(2\pi) = \frac{a_0}{2}$$

But what about all the other integrals, how will all those other integrals contribute to the value of $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ on the left hand side of (2)? We can use some of our newly developed *Mathematica* muscles to determine this quickly:

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In[1]:= Integrate[Cos[n x], {x, -π, π}, Assumptions → Element[n, Integers]]

Out[1]= 0

In[2]:= Integrate[Sin[n x], {x, -π, π}, Assumptions → Element[n, Integers]]

Out[2]= 0
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(We could have actually done both integrations at once :

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In[3]:= Integrate[{Cos[n x], Sin[n x]}, {x, -π, π}, Assumptions → Element[n, Integers]]

Out[3]= {0, 0}
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The result in output line[3] means that the set of answers returned by the integral is {0, 0}, indicating that both integrals yielded a value of zero over this interval.)

This result is critical; by integrating each term between -π and π we have that all but one term goes to zero; this means that the very complicated looking equation (2) has only one non-zero term on the right. Equation (2) then becomes:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \frac{a_0}{2} \Rightarrow$$

$$\mathbf{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx} \tag{3}$$

Now, how do we find the values of the other coefficients, say like a_1? We would like to find a technique that would cause all terms except the a_1 term on the right hand side of (1) to vanish. If we can find such a method, we would have a simple way of equating a_1 to f(x).

Let's see what happens if we multiply each term in (1) by cos(x)/2π, and then integrate all terms from -π to π:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = {} & \frac{1}{2\pi}\frac{a_0}{2}\int_{-\pi}^{\pi}\cos x\,dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\cos(x)\,dx + \\ & a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\cos(2x)\,dx + a_3\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\cos(3x)\,dx + \ldots \\ & + b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\sin(x)\,dx + b_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\sin(2x)\,dx + b_3\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\sin(3x)\,dx + \ldots\end{aligned} \tag{4}$$

Look carefully at each integral in eq. (4). The first set of integrals all involve terms of cos (x)·cos (n x) or cos (x)·sin (n x). From our previous work with these trig integrals, we expect that the only non zero integral is the second one on the right side of (4).

Explicitly evaluating integrals, we find :

```
In[5]:= Integrate[{Cos[x], Cos[x] Cos[x], Cos[x] Cos[n x], Cos[x] Sin[n x]},
          {x, -π, π}, Assumptions → Element[n, Integers]]

Out[5]= {0, π, 0, 0}
```

In the computation above, we simultaneously evaluated four types of integrals over the limits indicated. The output line[5] tells us that all these integrals are zero except for the second one. In other words, equation (4) reduces to :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos x \, dx = \frac{a_1}{2\pi}\pi \Rightarrow$$

$$\mathbf{a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos x \, dx} \tag{5}$$

We can generalize this technique and determine any coefficient a_n by multiplying every term in equation (1) by $\frac{1}{2\pi}$cos(nx) and then integrating from -π to π. Equation (1) becomes:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos(n x) \, dx =$$

$$\frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx) \, dx + \frac{1}{2\pi}\sum_{m=1}^{\infty}\left[a_m \int_{-\pi}^{\pi} \cos(m x) \cos(n x) \, dx + b_m \int_{-\pi}^{\pi} \cos(n x) \;\; \sin(m x)\right] \tag{6}$$

With all our practice in looking at these integrals, expression (6) should not appear quite so daunting now. The orthogonality property of cos and sin functions determines that for any particular value of n, every integral but one will vanish on the right hand side of (6). The only integral that will non - zero will be the integral where the value of the index m equals the value of n. Therefore, equation (6) tells us :

```
In[6]:= Integrate[{Cos[n x], Cos[m x] Cos[n x], Cos[n x] Sin[m x]},
          {x, -π, π}, Assumptions → Element[{m, n}, Integers]]

Out[6]= {0, 0, 0}
```

that all of the integrals in (6) are zero when m does not equal n, but when n = m,

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In[7]:= Integrate[Cos[n x] Cos[m x], {x, -π, π}, Assumptions → Element[{m, n}, Integers] && m == n]

Out[7]= π
```

Thus, the only non-zero term on the right side of (6) occurs when m=n, and that term is:

$$\frac{1}{2\pi} a_n \times \pi = \frac{a_n}{2}$$

This very powerful result allows us to calculate any coefficient, a_n:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos(n x) \, dx = \frac{a_n}{2} \Rightarrow$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n x)\, dx \qquad (7)$$

To find the coefficients b_n, we employ the same technique, except this time multiplying each term in eq. (1) by sin(nx) and integrating over [-π,π]. This allows us to determine all of the coefficients b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n x)\, dx \qquad (8)$$

And now we are complete. Now we have the tools to write any function f (x) as a Fourier series as in equation (1), and evaluate all the necessary coefficients by using equations (3), (7) and (8)

We can write functions as a series of sines and cosines because of the orthogonal nature of these trig functions. We will see that there are many other sets of orthogonal functions that can be used as a basis set to expand functions. For the next two weeks, we will focus on Fourier expansions. In April, we will study the properties of Legendre polynomials and show that they constitute a complete set of orthogonal functions. We will further learn that we can write any "reasonably well-behaved" function in terms of a series of Legendre polynomials.

In our next episode, we will see how to calculate these coefficients and produce Fourier series.