PHYSICS 301
FIRST HOUR EXAM
28 FEB. 2011

This is a closed book, closed note exam. Calculators will not be needed or permitted on this exam. Do all your writing in your blue books, making sure your name is on each blue book you use. You can do questions in any order as long as you clearly indicate the number of the question you are solving. Solutions must be complete, clear and correct to receive full credit. You may solve problems using any techniques at your disposal unless (as in Question 1) a specific approach is required. The point value of each question is indicated in parentheses.

1. Use Einstein summation notation only to determine which of the following expressions is equal to $\nabla\times(\mathbf{A}\times\mathbf{B})$:

a) $(\mathbf{B}\cdot\nabla)\,\mathbf{A} + (\mathbf{A}\cdot\nabla)\,\mathbf{B} - \mathbf{A}\,(\nabla\cdot\mathbf{B}) + \mathbf{B}\,(\nabla\cdot\mathbf{A})$

b) $(\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla\cdot\mathbf{B}) - \mathbf{B}\,(\nabla\cdot\mathbf{A})$

c) $(\mathbf{B}\cdot\nabla)\,\mathbf{A} + (\mathrm{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla\cdot\mathbf{B}) + \mathbf{B}\,(\nabla\cdot\mathbf{A})$

Solution :

We follow what is by now a pretty standard pattern in approaching summation notation proofs:

$$\nabla\times(\mathrm{A}\times\mathrm{B}) \to \epsilon_{mni}\,\frac{\partial}{\partial x_n}\left(\epsilon_{ijk}\,A_j\,B_k\right) =$$

$$\epsilon_{imn}\,\epsilon_{ijk}\,\frac{\partial}{\partial x_n}\,A_j\,B = \left(\delta_{mj}\,\delta_{nk} - \delta_{mk}\,\delta_{nj}\right)\left(\frac{\partial}{\partial x_n}\left(A_j\,B_k\right)\right) =$$

$$\delta_{mj}\,\delta_{nk}\left(\frac{\partial}{\partial x_n}\left(A_j\,B_k\right)\right) - \delta_{mk}\,\delta_{nj}\left(\frac{\partial}{\partial x_n}\left(A_j\,B_k\right)\right)$$

setting j = m and k = n in the first parentheses, and k = m and j = n in the second :

$$\frac{\partial}{\partial x_n}(A_m B_n) - \frac{\partial}{\partial x_n}(A_n B_m) =$$
$$B_n \frac{\partial}{\partial x_n} A_m + A_m \frac{\partial}{\partial x_n} B_n - A_n \frac{\partial}{\partial x_n} B_m - B_m \frac{\partial}{\partial x_n} A_n$$

The terms onthe right hand side of eq. (1) become :

$$(\mathbf{B} \cdot \nabla)\,\mathbf{A} + \mathbf{A}\,(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\,\mathbf{B} - \mathbf{B}\,(\nabla \cdot \mathbf{A})$$

and this is equivalent to option B above.

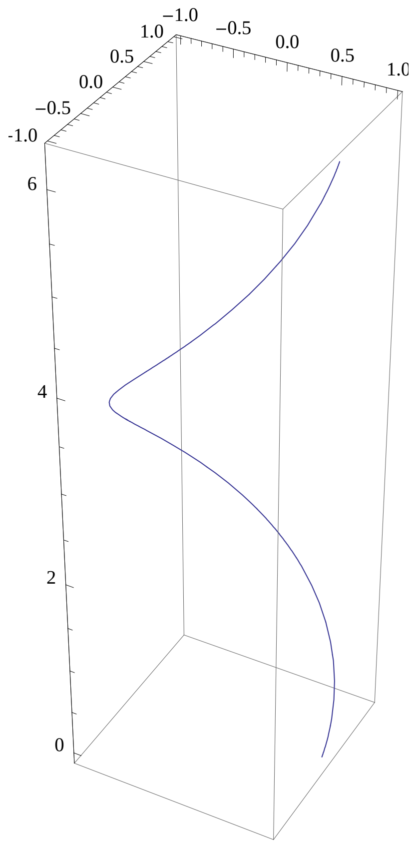
(Use Einstein summation notation to expand the original expression and determine which of the three choices is correct). (20)

2. Consider the scalar :

$$\phi = y^2 \sin x + x z^3 - 4 y + 2 z$$

a) Find the vector valued function **F** such that $\mathbf{F} = \nabla\phi$. (10)

b) Find the work done by **F** in moving a particle from (0, 0, 0) to (1, 0, 2 π) along the helical path parameterized as {cos t, sin t, t}. A graph of this path is shown below : (10)



Solution :

The only calculus you need to do for this question is in determining the gradient :

$$\mathbf{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}} = \left(y^2 \cos z + z^3\right)\hat{\mathbf{x}} + (2\,y\sin x - 4)\,\hat{\mathbf{y}} + \left(3\,x\,z^2 + 2\right)\hat{\mathbf{z}}$$

Now, your instincts should tell you that if you are asked to find a line integral over a fairly ugly path, you might first want to think if there is an easier way to approach this. And many of you thought, aha, I will take the curl of F and see if it is zero (it is), and therefore, the work done by F is independent of path. Now, you did not actually need to take the curl (if you do, you should find it is zero), since ... you derived **F** from a scalar potential; and if **F** is derivable from a scalar potential, then we already know it is conservative and the line integral is path independent. Moreover, if the force is derived from a scalar, we know that the line integral is simply :

$$W = \int_C \mathbf{F}\cdot d\mathbf{l} = \int_C \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{z}} \right)\cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) = \int_{(0,0,0)}^{(1,0,2\pi)} d\phi$$

$$= \phi(1, 0, 2\pi) - \phi(0, 0, 0) = (2\pi)^3 + 2(2\pi)$$

Let' s see if this reproduces the answer we get doing the explicit line integral. Since the force must be conservative (it is derived from a scalar potential, remember), we know the work done is path independent, so we can choose any path, say the path from (0, 0, 0) to (1, 0, 0) followed by the path from (1, 0, 0) to (1, 0, 2 π). If we take this path, we need to integrate $\mathbf{F}\cdot d\mathbf{l}$ along the + x axis, and then along the + z axis, this gives us :

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^1 F_x\,dx + \int_0^{2\pi} F_z\,dz = \int_0^1 \left(y^2 \cos z + z^3\right) dx + \int_0^{2\pi} \left(3\,xz^2 + 2\right) dz$$

Along the x axis, y = z = 0, so the first integral is zero; along the z axis, x = 1 and we have :

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_0^{2\pi} \left(3\,z^2 + 2\right) dz = (2\pi)^3 + 2(2\pi) \text{ as before.}$$

If we travel along the straight line from (0, 0, 0) to (1, 0, 2 π), we are traversing the path described by :

$$z = 2\pi x$$

so if we set x = t our parameterizations become :

$$x = t;\; dx = dt$$
$$z = 2\pi t;\; dz = 2\pi\,dt$$

and our line integral becomes (there will be no integral over y since y = 0 everywhere along the path) :

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int F_x\,dx + \int F_z\,dz = \int_0^1 (2\pi t)^3\,dt + \int_0^1 \left(3\,t\,(2\pi t)^2 + 2\right)(2\pi\,dt)$$

$$= 2\pi^3 + 6\pi^3 + 2(2\pi) \text{ as before.}$$

3. Find the value of

$$\int_S \mathbf{F} \cdot \mathbf{n}\, da$$

where **F** is the function :

$$\mathbf{F} = \{4\,x\,z,\ -y^2,\ y\,z\}$$

and S is the surface of the unit cube in the first octant (in other words, a cube of length one bounded by the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1). (20)

Solution :

The easiest way to approach this problem is to use the divergence theorem:

$$\int_S \mathbf{v} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{v}\, d\tau$$

Taking the divergence of **F**, we get :

$$\int_V \nabla \cdot \mathbf{F}\, d\tau = \int_V (4\,z - 2\,y + y)\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^1 (4\,z - y)\, dx\, dy\, dz$$

This is a very straight - forward volume integral whose value is 3/2. We can verify using Mathematica :

```
In[104]:= Clear[x, y, z]
          Integrate[(4 y - z), {z, 0, 1}, {y, 0, 1}, {x, 0, 1}]

Out[105]= 3
          -
          2
```

If we solved this via direct integration of the surface integral, we would break the total surface integral into six separate integrals (one for each face of the cube). Let' s identify these integrals as :

- I1 : where the normal is the + x direction, x=1, and the integral is over dy dz
- I2 : where the normal is in the **-** x direction and the integral is zero since x = 0
- I3: where the normal is in the +y direction,y = 1, and the integral is over dx dz
- I4: where the normal is in the **-**y direction, y=0 so the integral is zero
- I5: where the normal is in the +z direction, z=1 and the integral is over dx dy
- I6: where the normal is in the **-**z direction and z=0, so the integral is zero

The three non - zero integrals, I1, I3 and I5 give us :

$$\int_S \mathbf{v}\cdot\mathbf{n}\, da = \int_0^1\int_0^1 4\,x\,z\,dy\,dz + \int_0^1\int_0^1 -y^2\,dx\,dz + \int_0^1\int_0^1 yz\,dx\,dy$$

Remembering that x = 1 in I1, y = 1 in I3 and z = 1 in I5, we get :

$$\int_0^1\int_0^1 4\,z\,dy\,dz + \int_0^1\int_0^1 -(1)^2\,dx\,dz + \int_0^1\int_0^1 y\,dx\,dy = 2 - 1 + \frac{1}{2} = \frac{3}{2}$$

4. The two dimensional parabolic coordinate system is defined by the transformation equations :

$$x = \frac{1}{2}\left(u^2 - v^2\right) \text{ and } y = u\,v$$

Derive the unit vectors for this system and determine whether it is an orthogonal system. (10)

Solution :

To find the unit vectors in the parabolic coordinate system, we recall:

$$\hat{\mathbf{u}} = \frac{\frac{\partial \mathbf{r}}{\partial u}}{\left|\frac{\partial \mathbf{r}}{\partial u}\right|} \text{ and } \hat{\mathbf{v}} = \frac{\frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial v}\right|}$$

we write the position vector, r as :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} = \frac{1}{2}\left(u^2 - v^2\right)\hat{\mathbf{x}} + u\,v\,\hat{\mathbf{y}}$$

where we make use of the transformation equations. Then, performing the indicated partial differentiations with respect to u and v, we get :

$$\hat{\mathbf{u}} = \frac{u\,\hat{\mathbf{x}} + v\,\hat{\mathbf{y}}}{\sqrt{u^2 + v^2}};\ \hat{\mathbf{v}} = \frac{-v\,\hat{\mathbf{x}} + u\,\hat{\mathbf{y}}}{\sqrt{u^2 + v^2}}$$

Straight - forward calculation of **u·v** shows it is zero; and **u·u = v·v** = 1 proving this is an orthogonal coordinate system.

5. Calculate the curl of the vector field :

$$\mathbf{F} = \{2\,x\,y + z^3,\ x^2,\ 3\,x\,z^2\}$$

Compute the value of

$$\int_C \mathbf{F} \cdot d\mathbf{l}$$

where C is the regular hexagon whose center is at the origin. Assume the hexagon is lying in the x-y plane. (10)

Solution :

We make use of Stokes' Theorem to analyze this problem:

$$\int_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, da$$

The instincts that were alerted by the torturous contour in problem 2 should also be flaring up now; rather than try to parameterize the hexagon, Stokes' Theorem suggests that we take the curl of this function to see if it is zero. If it is zero, then we already know that the line integral is zero since the curl is zero; alternately, we know the line integral of a conservative function is zero on a closed path. So, curl away :

```
Needs["VectorAnalysis`"]
f = {2 x y + z^3, x^2, 3 x z^2};
Curl[f, Cartesian[x, y, z]]
```

```
{0, 0, 0}
```

Voila. Problem solved in entirety.

6. Write a short Mathematica program that will produce the following output : For each number k between 1 and 10, print twice the value of k is the number is prime, and half the value of k is the number is not prime. Spelling, spacing, syntax and capitalization all matter. (20)

If you use a Do loop;

```
Do[If[PrimeQ[k], Print[2 k], Print[k / 2]], {k, 10}]
```

$\frac{1}{2}$

4

6

2

10

3

14

4

$\frac{9}{2}$

5

Using a For loop;

```
For[k = 1, k < 11, k++, If[PrimeQ[k], Print[2 k], Print[k / 2]]]
```

$\frac{1}{2}$

4

6

2

10

3

14

4

$\frac{9}{2}$

5

Using a While loop;

```
Clear[k]
k = 1; While[k < 11, If[PrimeQ[k], Print[2 k], Print[k / 2]]; k++]
```

$\frac{1}{2}$

4

6

2

10

3

14

4

$\frac{9}{2}$

5

USEFUL INFORMATON AND FORMULAE

$$\nabla = \left(\frac{\partial}{\partial x}\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial y}\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial z}\right)\hat{\mathbf{z}}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{l}$$

$$\int_S \mathbf{v} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{v}\, d\tau$$

$$\int_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, da$$

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

$$r = \left|\mathbf{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

$$\epsilon_{ijk}\,\epsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp}$$