FORMULAE , RESULTS AND USEFUL EXPRESSIONS
SECOND HOUR EXAM

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\sin(n x) = \frac{e^{i n x} - e^{-i n x}}{2 i} \qquad \cos(n x) = \frac{e^{i n x} + e^{-i n x}}{2}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n x} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i n x}\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n \pi x/L} \qquad c_n = \frac{1}{2L}\int_{-\pi}^{\pi} f(x)\, e^{-i n \pi x/L}\,dx$$

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \sum_{n=1}^{\infty} n\, a_n x^{n-1} \Rightarrow \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

$$\left(1 - x^2\right) y'' - 2 x y' + m(m+1) y = 0$$

$$g(x, h) = \frac{1}{\sqrt{1 + h^2 - 2 h x}} = \sum_{m=0}^{\infty} P_m(x)\, h^m$$

$$c^2 = a^2 + b^2 - 2 a b \cos\theta$$

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x) \qquad c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\, P_m(x)\,dx$$

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right);$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right);\; P_4(x) = \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1}\,\delta_{mn}$$

The exam might require the evaluation of definite integrals. I will assume the ability to integrate polynomials, and present below the results of a series of indefinite integrations by parts. If one or more of these integrals appear on the exam, you will need to recognize which integral is relevant, and you will need to evaluate it at the appropriate limits.

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In[1]:= Integrate[{x Cos[n x], x Sin[n x]}, x]
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Out[1]= $\left\{\frac{\text{Cos}[n\,x]}{n^2} + \frac{x\,\text{Sin}[n\,x]}{n},\; -\frac{x\,\text{Cos}[n\,x]}{n} + \frac{\text{Sin}[n\,x]}{n^2}\right\}$

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In[2]:= Integrate[{x Cos[n π x / L], x Sin[n π x / L]}, x]
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Out[2]= $\left\{\frac{L^2\,\text{Cos}\left[\frac{n\pi x}{L}\right]}{n^2\pi^2} + \frac{L\,x\,\text{Sin}\left[\frac{n\pi x}{L}\right]}{n\pi},\; -\frac{L\,x\,\text{Cos}\left[\frac{n\pi x}{L}\right]}{n\pi} + \frac{L^2\,\text{Sin}\left[\frac{n\pi x}{L}\right]}{n^2\pi^2}\right\}$

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In[3]:= Integrate[{x^2 Cos[n x], x^2 Sin[n x]}, x]
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Out[3]= $\left\{\frac{2\,x\,\text{Cos}[n\,x]}{n^2} + \frac{\left(-2 + n^2 x^2\right)\text{Sin}[n\,x]}{n^3},\; -\frac{\left(-2 + n^2 x^2\right)\text{Cos}[n\,x]}{n^3} + \frac{2\,x\,\text{Sin}[n\,x]}{n^2}\right\}$

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In[4]:= Integrate[{x^2 Cos[n π x / L], x^2 Sin[n π x / L]}, x]
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Out[4]= $\left\{\frac{L\left(2\,L\,n\,\pi\,x\,\text{Cos}\left[\frac{n\pi x}{L}\right] + \left(-2L^2 + n^2\pi^2x^2\right)\text{Sin}\left[\frac{n\pi x}{L}\right]\right)}{n^3\pi^3},\right.$
$\left.\frac{L\left(\left(2L^2 - n^2\pi^2x^2\right)\text{Cos}\left[\frac{n\pi x}{L}\right] + 2\,L\,n\,\pi\,x\,\text{Sin}\left[\frac{n\pi x}{L}\right]\right)}{n^3\pi^3}\right\}$

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In[5]:= Integrate[{x^3 Cos[n x], x^3 Sin[n x]}, x]
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Out[5]= $\left\{\frac{3\left(-2 + n^2x^2\right)\text{Cos}[n\,x]}{n^4} + \frac{x\left(-6 + n^2x^2\right)\text{Sin}[n\,x]}{n^3},\; -\frac{x\left(-6 + n^2x^2\right)\text{Cos}[n\,x]}{n^3} + \frac{3\left(-2 + n^2x^2\right)\text{Sin}[n\,x]}{n^4}\right\}$

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In[6]:= Integrate[{x^3 Cos[n π x / L], x^3 Sin[n π x / L]}, x]
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Out[6]= $\left\{\frac{L\left(\left(-6L^3 + 3\,L\,n^2\pi^2x^2\right)\text{Cos}\left[\frac{n\pi x}{L}\right] + n\,\pi\,x\left(-6L^2 + n^2\pi^2x^2\right)\text{Sin}\left[\frac{n\pi x}{L}\right]\right)}{n^4\pi^4},\right.$
$\left.\frac{L\left(\left(6L^2\,n\,\pi\,x - n^3\pi^3x^3\right)\text{Cos}\left[\frac{n\pi x}{L}\right] + 3\,L\left(-2L^2 + n^2\pi^2x^2\right)\text{Sin}\left[\frac{n\pi x}{L}\right]\right)}{n^4\pi^4}\right\}$