PHYS 301
FIRST HOUR EXAM
SPRING 2012

This is a closed book, closed note exam. Turn off and store out of sight all electronic devices for the duration of the exam. Do all your writing in your blue book(s) making sure you put your name on each book you use. You may do questions in any order (just remember to label which question you are answering). **All answers must show complete and clear solutions.** The value of each question is represented by the numbers in parentheses. Please refer to the sheet of formulae and results as needed.

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1. a) Using only Einstein summation notation, determine which of the following identities is true :

א) $\nabla\times(f\,\mathbf{A}) = \mathbf{A}\times(\nabla f) - f\,(\nabla\times\mathbf{A})$

ב) $\nabla\times(f\,\mathbf{A}) = f\,(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

ג) $\nabla\times(f\,\mathbf{A}) = \mathbf{A}\times(\nabla f) + f\,(\nabla\times\mathbf{A})$

where f is a scalar and **A** is a vector.

Solution : The correct answer is ב which we show through standard use of summation notation :

$$\nabla\times(f\,\mathbf{A}) \rightarrow \epsilon_{ijk}\frac{\partial}{\partial x_j}(f\,A_k) = f\,\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k + \epsilon_{ijk}A_k\frac{\partial}{\partial x_j}f$$

The first term on the right is f $\nabla\times\mathbf{A}$, and the second term on the right involves the curl of A with grad f; the sign of this term is negative since the permutation is anti - cyclic (since the permuation tensor has the order "ijk" and the subscripts of A and the del operator is "kj". Therefore we have :

$$\nabla\times(f\,\mathbf{A}) \rightarrow \epsilon_{ijk}\frac{\partial}{\partial x_j}(f\,A_k) = f\,\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k + \epsilon_{ijk}A_k\frac{\partial}{\partial x_j}f = f\,(\nabla\times\mathbf{A}) - \mathbf{A}\times\nabla f.$$

b) Use the correct identity in part a) to show that :

$$\int_S \nabla T\times d\mathbf{a} = -\oint_C T\,d\mathbf{l}$$

Hint : let **v** = **c**T in Stokes' Theorem where **c** is a constant vector and T is a scalar.(15 points for part a); 10 points for part b))

Solution : We begin by writing Stokes' Theorem where our vector is c T

$$\int_S \nabla \times (\mathbf{c}\,T) \cdot d\mathbf{a} = \int_C (\mathbf{c}\,T) \cdot d\mathbf{l}$$

We use the identity above to expand the integrand on the left hand side to :

$$\nabla \times (\mathbf{c}T) = T\,\nabla \times \mathbf{c} - \mathbf{c} \times \nabla T = -\mathbf{c} \times \nabla T$$

The last equality follows since c is a constant vector and the curl of a constant vector is zero. Stokes' theorem now becomes :

$$\int_S -(\mathbf{c} \times \nabla T) \cdot d\mathbf{a} = \int_C T\,\mathbf{c} \cdot d\mathbf{l}$$

The integrand on the left can be permuted to cyclically to become :

$$\int_S -\mathbf{c} \cdot (\nabla T \times d\mathbf{a}) = \int_C T\,\mathbf{c} \cdot d\mathbf{l}$$

Multiply both sides by c dot and divide by c^2 to produce:

$$\int_S \nabla T \times d\mathbf{a} = -\int_C T\,d\mathbf{l}$$

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2. Consider the function given by :

$$\mathbf{F} = y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}$$

a) Is this a conservative function? (Make sure you show work and explain your answer)

b) Compute the work done by this force along the closed loop contour in the x - y plane described by (0, 0) → (1, 0) → (1, 1) → (0, 0).

(10 pts for part a); 15 pts for part b))

Solutions :

a) Elementary methods allow you to determine that the curl of this vector is - 2. Since the curl is non zero, this is not a conservative force.

b) You can solve this either by taking the line integral directly or by using Stokes' Theorem. The work done by the force around the loop is :

$$W = \int \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

We know the curl of the vector is - 2, and the region is a right triangle whose base is 1 and whose height is 1, therefore the area of the triangle is 1/2, and the surface integral of the curl is just (-2 x 1/2) = -1. In other words:

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = -2 \int_S da = -1$$

We can also find this result by solving the line integral directly. The line integral is :

$$W = \int_C \mathbf{F} \cdot d\mathbf{l} = \int_C \left(F_x\, dx + F_y\, dy\right) = \int_C (y\, dx - x\, dy)$$

Along the first segment from (0, 0) to (1, 0), y = 0 and dy = 0, so the contribution to the total line integral is zero. Along the second segment from (1, 0) to (1, 1), x = 1 and dx = 0, so the integral becomes :

$$\int_0^1 - x\, dy = - \int_0^1 dy = - 1$$

Now, we already know the total line integral around the loop is - 1, and the contributions to the line integral from the first two segments equals - 1. This suggests strongly that the line integral along the diagonal line defined by y = x will be zero. We show this by parameterizing :

$y = x = t$

$dy = dx = dt$

Then, the line integral becomes :

$$\int_C \left(F_x\, dx + F_y\, dy\right) = \int_C (y\, dx - x\, dy) = \int_1^0 (t\, dt - t\, dt) = 0$$

And the line integral around the closed contour is - 1, just as we found computing the area integral.

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3. In plane polar coordinates, the transformation equations are :

$$x = \rho \cos\phi \quad y = \rho \sin\phi$$

and the scale factors are :

$$h_1 = h_\rho = 1 \quad h_2 = h_\phi = \rho$$

the position vector in this coordinate system is written as :

$$\mathbf{r} = \rho\, \hat{\boldsymbol{\rho}}$$

a) Find the unit vectors for $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$.

b) Find expressions for the time derivatives of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$.

c) Find the expressions for velocity and acceleration in the plane polar coordinate system.
(5 pts for part a), 10 pts for part b), 15 pomts for part c))

Solutions: The solutions for these are worked out in detail in the classnote “Orthogonal Curvilinear Coordinates.”

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4. Write a short *Mathematica* program that will compute and print the result obtained by adding the value of all prime numbers between 1 and 100 and subtracting the value of all non prime numbers in that interval. In other words, the result of:

-1 + 2 + 3 - 4 + 5 - 6 ... .. + 97 - 98 - 99 - 100 (1 is not a prime number).

Solution:

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Clear[result]
result=0;
Do[If[PrimeQ[n], result=result+n, result=result-n],{n,100}]
Print[result]
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-2930
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Extra Credit (5 pts) : The following phrase is used as a mnemonic (memory aiding device) to remember the first seven prime numbers :

In the early morning, astronomers spiritualized nonmathematicians.

Explain how does this phrase represents the first seven primes.

Solution: The number of letters in each word represents the next prime number. There are 2 letters in the first word, 3 in the second, 5 in the third, and so on.

USEFUL INFORMATION AND FORMULAE

$$\nabla = \left(\frac{\partial}{\partial x}\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial y}\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial z}\right)\hat{\mathbf{z}}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{l}$$

$$\int_S \mathbf{v} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{v}\, d\tau$$

$$\int_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$$

$$r = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$