PHYS 301
SECOND HOUR EXAM - 2012
SOLUTIONS

1. We are asked to find the Fourier trig series for the function :

$$f(x) = x^2, \ -\frac{1}{2} < x < \frac{1}{2}$$

This function is 2 L periodic on the interval [-1/2, 1/2] (so that L = 1/2) and is an even function. Therefore, symmetry allows us to write :

$$a_0 = \frac{2}{(1/2)} \int_0^{1/2} x^2\, dx; \quad a_n = \frac{2}{(1/2)} \int_0^{1/2} x^2 \cos\left(\frac{n\pi x}{(1/2)}\right) dx$$

$$b_n = \frac{1}{(1/2)} \int_{-1/2}^{1/2} x^2 \sin\left(\frac{n\pi x}{(1/2)}\right) dx = 0$$

We can use symmetry to show that the b' s are zero since the integrand is odd and integrated between - 1/2 and + 1/2. Computing the other coefficients :

$$a_0 = 4 \int_0^{1/2} x^2\, dx = 4\left(\frac{x^3}{3}\right)\Big|_0^{1/2} = \frac{1}{6}$$

To find an we use the data given on the sheet of results and forumlae :

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Integrate[x^2 Cos[n π x / L], x]
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$$\frac{L\left(2 L n \pi x \,\mathrm{Cos}\left[\frac{n\pi x}{L}\right] + \left(-2L^2 + n^2\pi^2 x^2\right)\mathrm{Sin}\left[\frac{n\pi x}{L}\right]\right)}{n^3 \pi^3}$$

The sin ($n\pi$ x/L) term goes to zero since sin ($n\pi$) = sin 0 = 0. The cos term at the lower limit is zero because of the factor of x; the cos term evaluated at x = 1/2 (and remember that L = 1/2) yields :

$$a_n = \frac{2}{(1/2)}\left[2\,(1/2)^2\,(1/2)\cos[n\pi]\right] \Big/ \left(n^2\pi^2\right) = \frac{(-1)^n}{n^2\pi^2}$$

And the Fourier series is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \mathrm{Cos}[2 n \pi x]}{n^2} = \frac{1}{12} - \left(\frac{\cos(2\pi x)}{1^2} - \frac{\cos(4\pi x)}{2^2} + \frac{\cos(3\pi x)}{3^2} + \ldots\right)$$

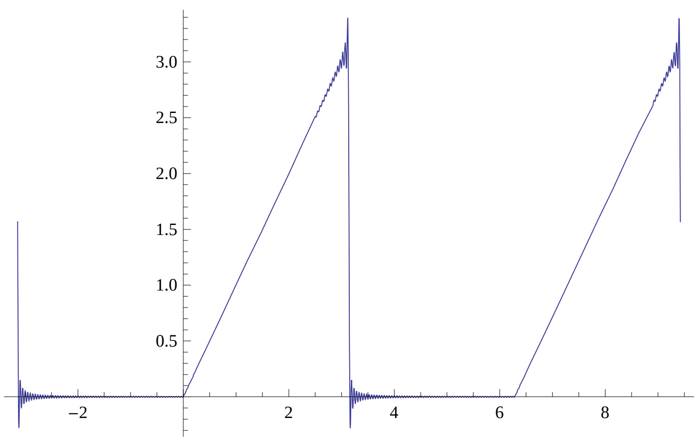
2. Our function in question is :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

whose Fourier series is :

$$f(x) = \frac{\pi}{4} - \left(\frac{2}{\pi}\right) \sum_{n,\,odd}^{\infty} \frac{\cos(n x)}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n x)}{n}$$

Two cycles of this series give us :



We know that the Fourier series converges to the function when the function is continuous, and converges to the midpoint at a discontinuity. The point in question, $x = \pi$ is a discontinuity, so the Fourier series converges to the midpoint of the discontinuity, or to $\pi/2$ when $x = \pi$. Now, substitute $x = \pi$ into the Fourier series and we obtain :

$$\frac{\pi}{2} = \frac{\pi}{4} - \left(\frac{2}{\pi}\right) \sum_{n,\,odd}^{\infty} \frac{\cos(n \pi)}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n \pi)}{n} \Rightarrow$$

$$\frac{\pi}{2} = \frac{\pi}{4} - \left(\frac{2}{\pi}\right) \sum_{n,\,odd}^{\infty} \frac{-1}{n^2} + 0$$

since $\cos(n\pi) = (-1)^n$ and this sum is over only the odd values of n, and $\sin(n\pi) = 0$ for all integer values of n.

Therefore, we have :

$$\frac{\pi}{4} = \frac{2}{\pi} \sum_{n,\,odd}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n,\,odd}^{\infty} \frac{1}{n^2} \text{ (as shown in Boas, problem 14, p. 358)}$$

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3. Start with our differential equation :

$$\left(x^2+1\right) y'' - 2\,x\,y' + 2\,y = 0$$

and substitute the trial solution; normal series solutions techniques yield :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n + \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - 2\sum_{n=1}^{\infty} n\,a_n\,x^n + 2\sum_{n=0}^{\infty} a_n\,x^n = 0$$

Re - index the second sum :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n + \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n - 2\sum_{n=1}^{\infty} n\,a_n\,x^n + 2\sum_{n=0}^{\infty} a_n\,x^n = 0$$

Strip out the n = 0 and n = 1 terms from the second and fourth sums, and the n = 1 term from the third sum :

$$2\,a_2 + 6\,a_3\,x - 2\,a_1\,x + 2\,a_0 + 2\,a_1\,x + \sum_{n=2}^{\infty}[n\,(n-1)\,a_n + (n+2)\,(n+1)\,a_{n+2} - 2\,n\,a_n + 2\,a_n]\,x^n = 0$$

The stripped out terms give us :

$$2\,a_2 + 2\,a_0 = 0 \Rightarrow a_2 = -a_0 \quad \text{and}\; a_3 = 0$$

We get no information about the $a_1$coefficient from the stripped out terms since the $a_1$terms sum to zero. Our recursion relation is then:

$$n\,(n-1)\,a_n + (n+2)\,(n+1)\,a_{n+2} - 2\,n\,a_n + 2\,a_n = 0 \Rightarrow$$

$$a_{n+2} = \frac{-[n\,(n-1) - 2\,n + 2]\,a_n}{(n+2)\,(n+1)} = \frac{-(n-2)\,(n-1)\,a_n}{(n+2)\,(n+1)}$$

We see from this form of the recursion relation that

$a_3 = 0$ since the right side goes to zero when $n = 1$;

$a_4 = 0$ since the right side goes to zero when $n = 2$.

Since every other coefficient is linked, we can see that all higher coefficients are zero, and thus the only non zero coefficients are a_0, a_1, and a_2. Thus, our series solution is:

$$\mathbf{y = a_0 + a_1\,x + a_2\,x^2 = a_0\left(1 - x^2\right) + a_1\,x}$$

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4. This is solved in detail in one of the posted classnotes. See the March 27 classnote "Legendre polynomials and their use in physics."

5. We can solve this using standard techniques for solving differential equations using Euler' s method. I will use y to represent the number of radioactive particles remaining in the nth iteration.

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Clear[y, n, λ, h]
y[0] = 1 000 000; t[0] = 0; h = 1; λ = 1.21 × 10^-4;
t[n_] := t[n] = t[n - 1] + h
y[n_] := y[n] = y[n - 1] - λ y[n - 1] h
Catch[Do[If[y[n] < y[0] / 2, Throw[{t[n], y[n]}]], {n, 20 000}]]
ListPlot[Table[{t[n], y[n]}, {n, 20 000}]]
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{5729, 499 948.}
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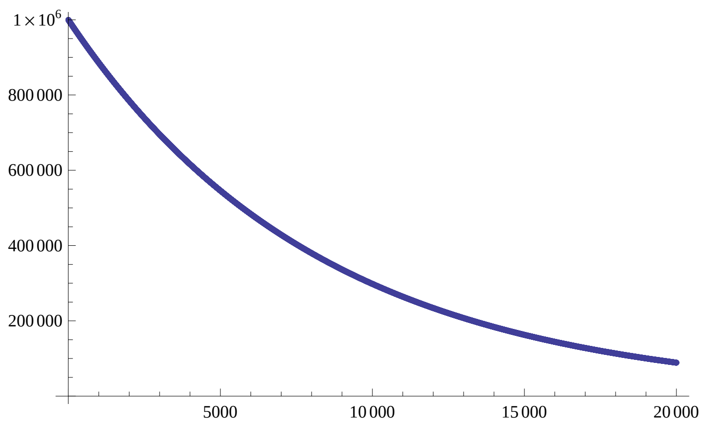

Since the half life of C - 14 is approximately 5730 years, I choose a step size of 1 year. This will give reasonable accuracy and will not require millions of iterations. y[0] is the initial abundance of nuclei; t[0] = 0 sets the clocks to zero as the decay begins; λ has the value cited.

The elapsed time of each iteration is defined recursively, and the number of remaining nuclei of C - 14, y[n], is defined in terms of y[n - 1] minus the number of nuclei that decayed since the last iteration.

The next statement terminates the Do loop once y[n] is less than 1/2 the initial value, and the output shows that we reproduce the half life of the isotope. The ListPlot plots 20, 000 years of data, to allow us to observe the decay through almost 4 half lives.