PHYS 301
First Hour Exam
Spring 2013

This is a closed book, closed note exam. You will not need nor be allowed to use calculators or other electronic devices on this test. Do all your writing in your blue book (s) making sure you put your name on each blue book you use. You may do questions in any order; please make sure to label clearly the question you are solving. Your answers must show clear and complete solutions; little to no credit will be given to "correct" answers that show no work. The numbers in parentheses indicate the value of the question.

1. Consider the function f (x) = Abs[x] on the interval (-2, 2). Compute the Fourier trig coefficients for this function, and write out explicitly the first three non zero terms of the Fourier series. You may use symmetry arguments where appropriate to determine Fourier coefficients. (30)

Solutions : First, recognize that Abs[x] is even on this interval, that means we can use symmetry to set the b coefficients equal to zero. Using symmetry, we can write the a coefficients as :

$$a_0 = \frac{2}{L}\int_0^L x\,dx = \frac{2}{2}\int_0^2 x\,dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 x\cos(n\pi x/2)\,dx = \left(\frac{4\cos(n\pi x/2)}{n^2\pi^2} + \frac{2x\sin(n\pi x/2)}{n\pi}\right)\Bigg|_0^2$$

The sin (n π x/2) terms are zero at x = 0 and 2; the coefficient then becomes :

$$a_n = \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos 0) = \frac{4}{n^2\pi^2}((-1)^n - 1) = \begin{cases} -8\big/n^2\pi^2, & \text{n odd} \\ 0. & \text{n even} \end{cases}$$

So that the Fourier series is :

$$f(x) = 1 - \frac{8}{n^2\pi^2}\left(\cos(\pi x/2) + \frac{\cos(3\pi x/2)}{3^2} + \frac{\cos(5\pi x/2)}{5^2}\right)$$

Many students did not use symmetry in computing the a_n coefficients; the correct way to solve for a_n is:

$$a_n = \frac{1}{2}\int_{-2}^0 (-x)\cos n x\,dx + \frac{1}{2}\int_0^2 x\cos n x\,dx$$

A common error was to forget that for - 2 < x < 0, | x | = -x; forgetting this sign led to an incorrect Fourier series.

2. Consider the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

The Fourier series for this function on $(-\pi, \pi)$ is :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

a) Use this series to show that :

$$\sum_{n=1,\,\text{odd}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} \qquad (5)$$

Solution : For part a) and b), the best way is to use Dirichlet' s theorem, not Parseval' s theorem as many students tried. Dirichlet' s theorem states that the Fourier series will converge to f where f is continuous. At x = 0, f is continuous and has the value f (0) = 0, therefore, applying Dirichlet' s theorem we get :

$$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \Rightarrow \frac{\pi^2}{8} = \sum_{n=1,\,\text{odd}}^{\infty} \frac{1}{n^2}$$

b) Use this series to show that :

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \qquad (10)$$

Solution. Here, evaluate the series at $x = \pi/2$; $f(\pi/2) = \pi/2$. When $x = \pi/2$, all the cos terms go to zero, and the sin terms go to ± 1 alternately. If you set $x = \pi/2$, you will obtain the series above.

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3. In class and for homework, you used Parseval' s theorem to evaluate sums of the form :

$$\sum_{n=1}^{\infty} \frac{1}{n^{\zeta}}$$

where ζ is an integer. One of the steps is to find the average value of the square of some function over an appropriately chosen interval. Suppose you want to find the sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^8}$$

by evaluating an appropriate function over $(-\pi, \pi)$. What function could you use in conjunction with Parseval' s Theorem to evaluate this sum? Explain why this function will allow you to compute the value of this sum. There are many possible choices, you just have to identify and justify one of them. Do not attempt to evaluate Fourier coefficients; do not attempt to actually evaluate the sum.

The question asks only to identify a function you could use, and explain why that function will allow you to find the sum. (btw, the sum converges to $\pi^8/9450$). (15)

Solution : Parseval' s theorem tells us :

$$\text{average of } f(x))^2 \text{ from } (-\pi.\pi) = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} a_n{}^2 + \sum_{n=1}^{\infty} b_n{}^2$$

In order to equate $\Sigma 1/n^8$ to a number, we need to generate a series of terms involving $1/n^8$. We know from Parseval's theorem that we will square the Fourier coefficients. Thus, in order to get a term involving $1/n^8$, we need to square a term involving n^4. We will obtain coefficients involving $1/n^4$ terms if we compute Fourier coefficients for f(x) = x^4 (or other quartic functions).

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4. Use Einstein summation notation to show that :

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) \quad (20)$$

Solution. This is worked out in detail in the epsilondelta class note:
http : // luc.edu/faculty/dslavsk/courses/phys301/epsilondelta.pdf

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5. Write a short Mathematica program that will, for each integer k from 1 to 100, print out the value of k/2 if the number is even and print out the value of 3 k + 1 if the number is odd. (20)

Solution : There are many possible ways to write this, a simple way is :

```
Do[If[EvenQ[k], Print[k / 2], Print[3 k + 1]], {k, 20}]
```

```
4
1
10
2
16
3
22
4
28
5
34
6
40
7
46
8
52
9
58
10
```

Your loop should be {k, 100}; I just truncated this after k = 20 to keep the list from getting too long.

EQUATIONS AND RESULTS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,\pi\,x/L) + \sum_{n=1}^{\infty} b_n \sin(n\,\pi\,x/L)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\,\pi\,x/L)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\,\pi\,x/L)\,dx$$

$$\text{average value of } (f(x))^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n{}^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n{}^2$$

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 0, & \text{any 2 indices equal} \\ 1, & \text{all indices different and cyclic} \\ -1, & \text{all indices different and anti} - \text{cyclic} \end{cases}$$

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

Values of indefinite integrals

```
Integrate[{x Cos[n x], x Sin[n x]}, x]
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$$\left\{\frac{\text{Cos}[n\,x]}{n^2} + \frac{x\,\text{Sin}[n\,x]}{n},\ -\frac{x\,\text{Cos}[n\,x]}{n} + \frac{\text{Sin}[n\,x]}{n^2}\right\}$$

```
Integrate[{x Cos[n π x / L], x Sin[n π x / L]}, x]
```

$$\left\{\frac{L^2\,\text{Cos}\left[\frac{n\,\pi\,x}{L}\right]}{n^2\,\pi^2} + \frac{L\,x\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]}{n\,\pi},\ -\frac{L\,x\,\text{Cos}\left[\frac{n\,\pi\,x}{L}\right]}{n\,\pi} + \frac{L^2\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]}{n^2\,\pi^2}\right\}$$

```
Integrate[{x^2 Cos[n x], x^2 Sin[n x]}, x]
```

$$\left\{\frac{2\,x\,\text{Cos}[n\,x]}{n^2} + \frac{\left(-2 + n^2\,x^2\right)\text{Sin}[n\,x]}{n^3},\ -\frac{\left(-2 + n^2\,x^2\right)\text{Cos}[n\,x]}{n^3} + \frac{2\,x\,\text{Sin}[n\,x]}{n^2}\right\}$$

Integrate[{x^2 Cos[n π x / L], x^2 Sin[n π x / L]}, x]

$$\left\{\frac{L\left(2\,L\,n\,\pi\,x\,\text{Cos}\left[\frac{n\,\pi\,x}{L}\right]+\left(-2\,L^2+n^2\,\pi^2\,x^2\right)\,\text{Sin}\left[\frac{n\,\pi\,x}{L}\right]\right)}{n^3\,\pi^3},\right.$$

$$\left.\frac{L\left(\left(2\,L^2-n^2\,x^2\,\pi^2\right)\,\text{Cos}\left[\frac{n\,x\,\pi}{L}\right]+2\,L\,n\,x\,\pi\,\text{Sin}\left[\frac{n\,x\,\pi}{L}\right]\right)}{n^3\,\pi^3}\right\}$$

Integrate[{x^3 Cos[n x], x^3 Sin[n x]}, x]

$$\left\{\frac{3\left(-2+n^2\,x^2\right)\,\text{Cos}[n\,x]}{n^4}+\frac{x\left(-6+n^2\,x^2\right)\,\text{Sin}[n\,x]}{n^3},\;-\frac{x\left(-6+n^2\,x^2\right)\,\text{Cos}[n\,x]}{n^3}+\frac{3\left(-2+n^2\,x^2\right)\,\text{Sin}[n\,x]}{n^4}\right\}$$

Integrate[{x^4 Cos[n x], x^4 Sin[n x]}, x]

$$\left\{\frac{4\,x\left(-6+n^2\,x^2\right)\,\text{Cos}[n\,x]}{n^4}+\frac{\left(24-12\,n^2\,x^2+n^4\,x^4\right)\,\text{Sin}[n\,x]}{n^5},\right.$$

$$\left.-\frac{\left(24-12\,n^2\,x^2+n^4\,x^4\right)\,\text{Cos}[n\,x]}{n^5}+\frac{4\,x\left(-6+n^2\,x^2\right)\,\text{Sin}[n\,x]}{n^4}\right\}$$