PHYS 301

SECOND HOUR EXAM--SOLUTIONS

This is a closed book, closed note exam. Calculators and other electronic devices will not be allowed. You may write on your exam sheets but must do all your work for grading in your blue book (s). You may do problems in any order; make sure your name is on each blue book you use. You must show complete solutions to receive full credit.

1. Consider the transformation equations between Cartesian and plane polar coordinates :

$$x = \rho \cos\phi \qquad y = \rho \sin\theta$$

In plane polar coordinates, the position vector is expressed as :

$$\mathbf{r} = \rho\, \hat{\boldsymbol{\rho}}$$

Find expressions for the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ and also expressions for the time derivatives of these unit vectors. Then find the expressions for velocity and acceleration in plane polar coordinates; your answers should be in terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. (30)

Solution: The solutions to this problem are detailed in the classnote of March 13 dealing with curvilinear orthogonal coordinates.

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2. Consider the differential equation :

$$y'' - x^2 y = 0$$

Assume a power series solution for this equation. Use series solution techniques to determine the recursion relation for this equation. Assuming that $a_0 = a_1 = 1$, write out the solution to the equation out to terms in x^9. (30)

Solution : We begin with the trial solution :

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting this into the original differential equation yields :

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Re - indexing the first sum by setting k = n - 2 and the second sum by setting k = n + 2, we obtain :

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=2}^{\infty} a_{n-2}\, x^n = 0$$

We can write both sums with the same limits by stripping out the n = 0 and n = 1 terms from the first sum and get :

$$2 a_2 + 6 a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)\, a_{n+2} - a_{n-2}]\, x^n = 0$$

For n ≥2, the recursion relation is :

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}$$

The stripped out terms tell us that $a_2 = a_3 = 0$. The recursion relation shows us that the $(n+2)^{nd}$ term is linked to the $(n-2)^{nd}$ term, so that we know each coefficient is related to the coefficient four units away. Thus:

$$a_6 = a_7 = a_{10} = a_{11} = 0$$

Since we are told that $a_0 = 1 = a_1$, we have:

$$a_4 = \frac{a_0}{4 \cdot 3} = \frac{1}{12}$$

$$a_5 = \frac{a_1}{5 \cdot 4} = \frac{1}{20}$$

$$a_8 = \frac{a_4}{8 \cdot 7} = \frac{a_0}{56 \cdot 12} = \frac{1}{672}$$

$$a_9 = \frac{a_5}{72} = \frac{a_1}{72 \cdot 20} = \frac{1}{1440}$$

and the series solution becomes :

$$y = 1 + x + \frac{x^4}{12} + \frac{x^5}{20} + \frac{x^8}{672} + \frac{x^9}{1440} + \ldots$$

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3. Consider a system of two charges. A charge of + q is located on the x axis at (a, 0). A charge of - q is located on the y - axis at (0, a). Determine the potential due to these charges at a point a distance r from the origin (such that r >> a). Express your result in terms of power series involving Legendre polynomials. Determine explicitly the leading non - zero term of the expansion. (20)

Solution : The potential at the given point will be the sum of potentials due to the individual charges. We have shown many times in class and classnotes that the potential due to the point charge at (a, 0) can be written as :

$$V_1 = \frac{k\,q}{r} \sum_{m=0}^{\infty} P_m(\cos\theta)\,(a/r)^m$$

The potential due to the negative charge at (0, a) can also be written in terms of a power series involving Legendre polynomials. Using the law of cosines, we can write the distance from the negative charge to the observation point as :

$$r_2 = \sqrt{1 + (a/r)^2 - 2\,a\,r\cos(90 - \theta)}$$

Since cos (90 - θ) = sin (90 - θ), we can express the potential due to the second charge as :

$$V_2 = -\frac{k\,q}{r} \sum_{m=0}^{\infty} P_m(\sin\theta)\,(a/r)^m$$

By superposition, the total potential is :

$$V = \frac{k\,q}{r} \sum_{m=0}^{\infty} (P_m(\cos\theta) - P_m(\sin\theta))\,(a/r)^m$$

The m = 0 term is zero, so the leading term in the expansion is the m = 1 term and is :

$$\frac{k\,q}{r}\left[(P_1(\cos\theta) - P_1(\sin\theta))\,(a/r)^1\right] = \frac{k\,q\,a}{r^2}(\cos\theta - \sin\theta)$$

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4. Consider an object at rest at a distance R_0 from the Earth such that $R_0 >> R_e$ (the radius of the Earth). The acceleration of the object will not be a constant, but will vary according to:

$$a = GM / r^2$$

where G is the Newtonian Gravitational Constant, M is the mass of the Earth, and r is the instantaneous distance between the object and the center of the Earth. For the given parameters below, write a short Mathematica program, using Euler' s method and recursion relations, to determine how long it will take the object to reach Earth (do not worry about details that the object will not actually travel all the way to the center of the Earth (the speed of the object on impact is approximately 10 km/s, so it would not take long to travel a distance equal to the radius of the Earth). For your parameters (all in MKS), use :

$G = 6.67 \times 10^{-11}$

$M = 6 \times 10^{27}$

$R_0 = 4 \times 10^{8}$

Assume the initial distance equals R_0 and the initial velocity is zero. There is no need (nor time) to provide commentary, and no need to produce plots. But show explicitly how the time is being computed (in other words, state clearly which variable or combination of variables will show the time to impact). (20)

In the code below, x is the distance from the Earth, a the acceleration, v is the velocity and h is the step size. Since I know the time will be a matter of days, I set h = 1 minute to avoid integrating every second. We obtain the result of an infall time of approximately 5 days. (The factor of 1440 is the number of minutes per day, so dividing nterms (the number of minutes) by minutes/day yields an answer in days.)

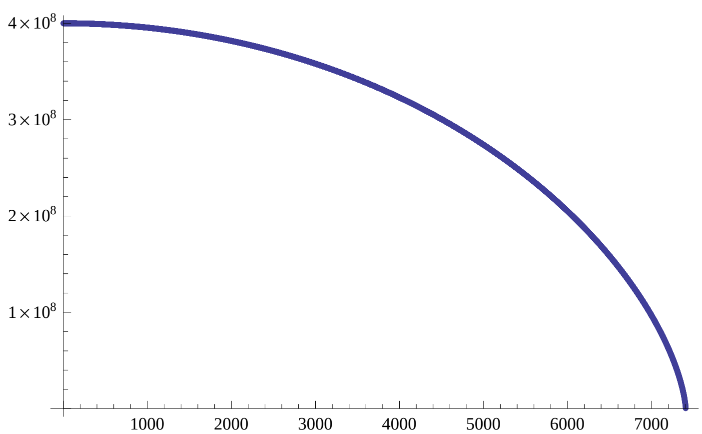
In[169]:=
```
Clear[a, x, newt, mass, v, h, nterms]
x[0] = 4 × 10^8; v[0] = 0;
newt = 6.67 × 10^(-11); mass = 6 × 10^24; h = 60;
a[x_] := a[x] = -newt mass / x^2
v[n_] := v[n] = v[n - 1] + h a[x[n - 1]]
x[n_] := x[n] = x[n - 1] + h v[n - 1]
nterms = Catch[Do[If[x[n] < 0, Throw[n - 1]], {n, 100000}]];
Print["Time to impact =", nterms / 1440 // N, " days"]
```

```
Time to impact =5.14306 days
```

We could write the code without an explicit definition of acceleration :

In[237]:=
```
Clear[x, v, h, newt, mass, nterms]
x[0] = 4 × 10^8; newt = 6.67 × 10^(-11); mass = 6 × 10^24; h = 60;
v[0] = 0;
v[n_] := v[n] = v[n - 1] - (h newt mass / x[n - 1]^2)
x[n_] := x[n] = x[n - 1] + h v[n - 1]
nterms = Catch[Do[If[x[n] < 0, Throw[n - 1]], {n, 100000}]];
ListPlot[Table[x[n], {n, nterms}]]
Print["Time to impact =", nterms / 1440 // N, " days"]
```

Out[243]=




```
Time to impact =5.14306 days
```

FORMULAE AND RESULTS

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

$$\hat{\mathbf{q}} = \frac{\frac{\partial \mathbf{r}}{\partial q}}{\left|\frac{\partial \mathbf{r}}{\partial q}\right|}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$g(x, h) = \frac{1}{\sqrt{1 + h^2 - 2hx}} = \sum_{m=0}^{\infty} P_m(x)\, h^m$$

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x) \qquad c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\, P_m(x)\, dx$$

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right);$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right); \quad P_4(x) = \frac{1}{8}\left(3 - 30x^2 + 35x^4\right)$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1}\,\delta_{mn}$$

$$\cos(90 - \theta) = \sin\theta$$