PHYS 301

SECOND HOUR EXAM SOLUTIONS--2014

1. We need to transform the coefficients and unit vectors into polar coordinates. We find the unit vectors by :

$$\hat{\boldsymbol{\rho}} = \frac{\partial \mathbf{r} / \partial \rho}{|\partial \mathbf{r} / \partial \rho|} = \frac{\cos\phi\, \hat{\mathbf{x}} + \sin \hat{\boldsymbol{\phi}}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\, \hat{\mathbf{x}} + \sin\phi\, \hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = \frac{\partial \mathbf{r} / \partial \phi}{|\partial \mathbf{r} / \partial \phi|} = \frac{-\rho\sin\phi\, \hat{\mathbf{x}} + \rho\cos\phi\, \hat{\mathbf{y}}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}$$

You can either solve these equations algebraically in terms of ρ and ϕ, or use matrix algebra techniques to determine that :

$$\hat{\mathbf{x}} = \cos\phi\, \hat{\boldsymbol{\rho}} - \sin\phi\, \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin\phi\, \hat{\boldsymbol{\rho}} + \cos\phi\, \hat{\boldsymbol{\phi}}$$

Now, using these expressions with our initial vector, we obtain :

$$\mathbf{v} = -\,\mathrm{y}\, \hat{\mathbf{x}} + \mathrm{x}\, \hat{\mathbf{y}} = -\rho\sin\phi\left(\cos\phi\, \hat{\boldsymbol{\rho}} - \sin\phi\, \hat{\boldsymbol{\phi}}\right) + \rho\cos\phi\left(\sin\phi\, \hat{\boldsymbol{\rho}} + \cos\phi\, \hat{\boldsymbol{\phi}}\right) =$$

$$(-\rho\sin\phi\cos\phi + \rho\cos\phi\sin\phi)\, \hat{\boldsymbol{\rho}} + \left(\rho\sin^2\phi + \rho\cos^2\phi\right) \hat{\boldsymbol{\phi}} = \rho\, \hat{\boldsymbol{\phi}}$$

Let' s think about what this answer means, and see if we can verify that meaning. When written in polar coordinates, the only non - zero component is in the $\boldsymbol{\phi}$ direction. The magnitude of the vector is ρ, indicating the magnitude increases outward from the origin. Let' s plot this vector to see if its behavior matches our result :

In[34]:=
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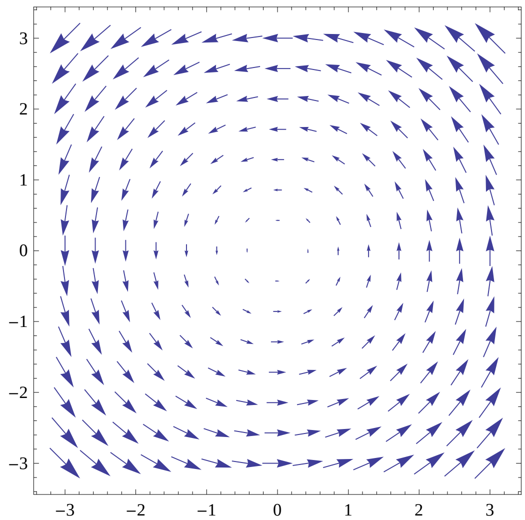
VectorPlot[{-y, x}, {x, -3, 3}, {y, -3, 3}]
```

Out[34]=



Note how the vectors form a circle at each radius, consistent with a vector that has only a $\boldsymbol{\phi}$ component. Also, it should be clear that the length of the vectors increases outward from the origin, exactly what we would expect from the vector $\rho\,\hat{\boldsymbol{\phi}}$.

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2. Apply standard series solutions techniques to :

$$y'' - 2\,x\,y' + k\,y = 0$$

to obtain :

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n - 2\sum_{n=1}^{\infty} a_n\,x^n + 2\,k\sum_{n=0}^{\infty} a_n\,x^n = 0$$

yields the recursion relation :

$$a_{n+2} = \frac{2\,(n-k)\,a_n}{(n+2)\,(n+1)}$$

Our first few even coefficients become :

$$a_2 = \frac{2\,(0-k)\,a_0}{2\cdot 1} = -k\,a_0$$

$$a_4 = \frac{2\,(2-k)\,a_2}{4\cdot 3} = \frac{-k\,(2-k)\,a_0}{6}$$

The odd branch has coefficients :

$$a_3 = \frac{2\,(1-k)\,a_1}{3\cdot 2} = \frac{(1-k)\,a_1}{3}$$

$$a_5 = \frac{2\,(3-k)\,a_3}{5\cdot 4} = \frac{2\,(3-k)\,(1-k)\,a_1}{5\cdot 4\cdot 3} = \frac{(3-k)\,(1-k)\,a_1}{30}$$

For initial conditions of $a_0 = a_1 = 1$, we have simply:

$$a_2 = -k;\;\; a_4 = -k\,(2-k)/6$$

$$a_3 = (1-k)/3;\ \ a_5 = (3-k)(1-k)/30$$

and our solutions are :

$$y = a_0\left[1 - k\,x^2 + \frac{(k-2)\,k}{6}\,x^4 + ...\right] + a_1\left[x + \frac{(1-k)}{3}\,x^3 + \frac{(3-k)(1-k)}{30}\,x^5 + ...\right]$$

b) Since our recursion relationship does not involve any derivatives of H, our first instinct should be to compute $\partial g/\partial t$. Taking the partial derivative with respect to t of both sides of the generating function yields :

$$\frac{\partial g}{\partial t} = 2\,(x-t)\exp\left(2\,x\,t - t^2\right) = \Sigma\, n\, H_n(x)\,\frac{t^{n-1}}{n!}$$

Now, notice that the exp function is simply the generating function, and the ratio (on the right) of n/n! = 1/(n - 1)! Thus, we have :

$$2\,x\,\Sigma\, H_n(x)\,\frac{t^n}{n!} - 2\,\Sigma\, H_n(x)\,\frac{t^{n+1}}{n!} = \Sigma\ H_n\,\frac{t^{n-1}}{(n-1)!}$$

Now we equate like terms (i.e., terms with the same exponent). If we want to compare all t^3 terms, we set n =3 in the first sum, n=2 in the second sum and n =4 in the sum on the right. Generalizing this allows us to write:

$$\frac{2\,x\,H_n(x)}{n!} - \frac{2\,H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}}{(n+1-1)!}$$

Multiply through by n! :

$$\boxed{\boldsymbol{2\,x\,H_n(x) - 2\,n\,H_{n-1}(x) = H_{n+1}(x)}}$$

(remember that in the second summation we get n!/(n - 1)! which is just n) and this recursion relation matches eq. 22.17 b on page 609 of the text.

c) Using this recursion relation allows us to find the values of H_3 and H_4:

if we set n = 2, we get :

$$H_3 = 2\,x\,H_2 - 2\cdot 2\,H_1 = 2\,x\left(4\,x^2 - 2\right) - 4\,(2\,x) = 8\,x^3 - 12\,x$$

if we set n = 3, we compute :

$$H_4 = 2\,x\,H_3 - 2\cdot 3\,H_2 = 2\,x\left(8\,x^3 - 12\,x\right) - 6\left(4\,x^2 - 2\right) = 16\,x^4 - 48\,x^2 + 12$$

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3. The purpose of this question was to move beyond ' grind and find' problems involving Stokes' Theorem, and examine the meaning of the theorem. Since we are dealing with the velocity field on the surface of the Earth (which represents our x - y plane), the normal to this plane is in the z direction. Therefore, the vorticity is simply the magnitude of the component of the curl of the v field perpendicular to the x - y plane. In other words, $\zeta = (\nabla\times\mathbf{v})\cdot\mathbf{n}$, so if we employ Stokes' Theorem:

$$\int_S (\nabla\times\mathbf{v})\cdot\mathbf{n}\,da = \int_C \mathbf{v}\cdot dl$$

we have simply that the integral on the left becomes $\iint \zeta\,dx\,dy = x\,y\,dx\,dy$, which is a trivial double integral to compute.

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4. Euler' s method allows us to write :

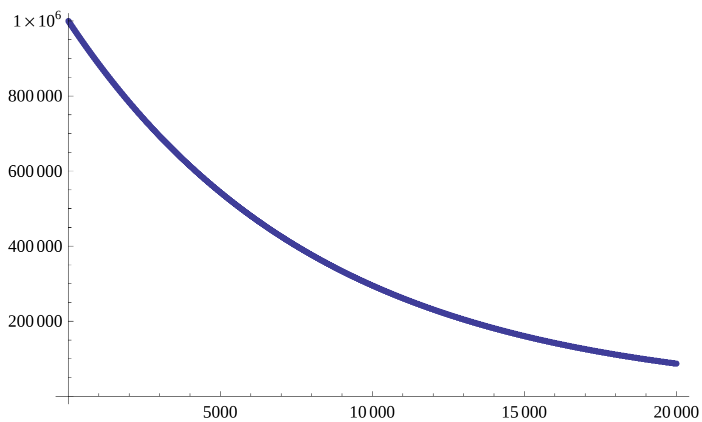
new value of a function = previous value of the function + (step - size) * derivative. For our case :

$$\frac{dN}{dt} = -\lambda N$$

we are given the derivative (and it equals - λ N). So, we can write our program as :

In[23]:=
```
Clear[nuclei, h, λ, nterms]
(* nuclei will represent the number of radioactive nuclei remaining at any given time t,
h is the stepsize, λ is the decay constant, for a typical case,
say that for Carbon -14, λ = 1.22 10^-4 yr^-1 *)
nuclei[0] = 10^6; λ = 1.22 × 10^(-4); h = 1;
(* setting h = 1 year is reasonable if we are dealing with a half life of 5730 years *)
nuclei[n_] := nuclei[n] = nuclei[n - 1] + h (-λ nuclei[n - 1])
ListPlot[Table[{n h, nuclei[n]}, {n, 1, 20 000}]]
```

Out[26]=



Where we have plotted results for a period of 20, 000 years, representing several half lives of decay. Now, let' s see how we can verify that the half - life of of this sample is 5730 years (i.e., in 5730 years, we should have 1/2 of the original sample remaining) :

In[21]:=
```
nterms = Catch[Do[If[nuclei[n] < nuclei[0] / 2, Throw[n - 1]], {n, 20 000}]];
Print["The half life of the sample is ", nterms, " years"]
```

```
The half life of the sample is 5681 years
```

Close enough, especially given that we used only 3 sig figs for λ. How much remains after two (and after three) half-lives? :

In[31]:=
```
Print["The fraction of the sample remaining after 2 half lives is ",
 nuclei[2 × 5730] / nuclei[0] // N,
 " The fraction of the sample remaining after three half lives is ",
 nuclei[3 × 5730] / nuclei[0]]
```

```
The fraction of the sample remaining after 2 half lives is 0.24704
  The fraction of the sample remaining after three half lives is 0.122787
```