PHYS 301
FIRST HOUR EXAM SOLUTIONS

1. We are given the product

$$\epsilon_{mnop}\,\epsilon_{mnop}$$

Since there are four repeated indices (m, n, o and p), this is equivalent to the quadruple sum :

$$\epsilon_{mnop}\,\epsilon_{mnop} = \sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{o=1}^{4}\sum_{p=1}^{4}\epsilon_{mnop}\,\epsilon_{mnop} = \epsilon_{1111}\,\epsilon_{1111} + \epsilon_{1112}\,\epsilon_{1112} + 254 \text{ more terms.}$$

Now, we don' t have to evaluate all these terms, since we know the only non - zero terms will be those in which the four indices are all different. Since there are 24 ways to permute four different numbers (1234, 1243, 1324, 1342, 1423, 1432 ...), there will be 24 non - zero terms. Since each term with four different indices is either + 1 or - 1, there will be 24 terms of either (1) (1) or (-1) (-1). Therefore, the value of this product is 24.

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2. We are given the function

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 3/2 \\ 0, & 3/2 < x < 2 \end{cases}$$

and we are told the function is periodic on (0, 2). This statement means explicitly that the function is defined to be 2 L periodic on (0, 2), so that the relevant value of L to use is L = 1, and our Fourier series will have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$$

with L = 1. (It is therefore incorrect to make any extension of this function since that will change the value of L and hence the periodicity of the function).

Using the relevant equations for computing Fourier coefficients, we have :

$$a_o = \frac{1}{L}\int_0^2 f(x)\,dx = \frac{1}{1}\int_1^{3/2} 1\cdot dx = \frac{1}{2}$$

$$a_n = 1\int_1^{3/2}\cos(n\pi x)\,dx = \frac{1}{n\pi}\sin(n\pi x)\Big|_1^{3/2} = \frac{1}{n\pi}(\sin(3n\pi/2) - 0)$$

$$b_n = 1\int_1^{3/2}\sin(n\pi x)\,dx = \frac{-1}{n\pi}\cos(n\pi x)\Big|_1^{3/2} = \frac{-1}{n\pi}(\cos(3n\pi/2) - (-1)^n)$$

From these coefficients we can conclude :

$$a_n = \frac{1}{n\pi}\begin{cases} 0, & \text{n even} \\ (-1)^{\frac{n+1}{2}}, & \text{n odd} \end{cases} \qquad b_n = \frac{1}{n\pi}\begin{cases} -1, & \text{n odd} \\ 2, & n = 2, 6, 10 \\ 0, & n = 4, 8, 12 \end{cases}$$

the Fourier expansion is then :

$$f(x) = \frac{1}{4} - \frac{1}{\pi}\left[\text{Cos}[\pi x] - \frac{\text{Cos}[3\pi x]}{3} + \frac{\text{Cos}[5\pi x]}{5} - \ldots\right] - \frac{1}{\pi}\left[\text{Sin}[\pi x] - \frac{2\,\text{Sin}[2\pi x]}{2} + \frac{\text{Sin}[3\pi x]}{3} - \ldots\right]$$

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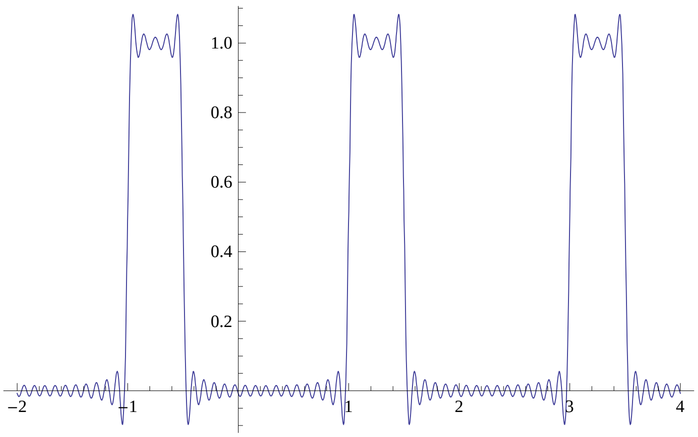
3. If we plot three cycles of this function :



We can see that the function is continuous at zero and f (0) = 0, and that the function has a discontinuity at x = 1. Dirichlet' s theorem tells us that the series will converge to the function where the function is continuous, and the series will converge to the midpoint of a discontinuity, therefore, Dirichlet' s theorem tells us that

f (0) = 1 and f (1) = 1/2 (the midpoint of the discontinuity). Now, if we set x = 0 in the Fourier series above we get :

$$f(0) = 0 = \frac{1}{4} - \frac{1}{\pi}\left[\cos 0 - \frac{\cos 0}{3} + \frac{\cos 0}{5} - \ldots\right] - \frac{1}{\pi}\left[\sin 0 - \sin 0 + \frac{\sin 0}{3} - \ldots\right]$$

$$= \frac{1}{4} - \frac{1}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right] \Rightarrow \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

as previously derived (see text p. 358)

If we set x = 1, then f (1) = 1/2, and :

$$f(1) = \frac{1}{2} = \frac{1}{4} - \frac{1}{\pi}\left[\cos(\pi) - \frac{\cos(3\pi)}{3} + \frac{\cos(5\pi)}{5} - \ldots\right] - \frac{1}{\pi}\left[\sin(\pi) - \sin(2\pi) + \frac{\sin(3\pi)}{3} + \ldots\right]$$

Since sin (n π) = 0 for integer values of n, and cos (n π) = -1 for odd values of n, we have :

$$\frac{1}{4} = \frac{-1}{\pi}\left[-1 + \frac{1}{3} - \frac{1}{5} - \ldots\right] \Rightarrow \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

as we found above. I have shown this choosing two suitable values of x, you only needed to choose one value.

4. The proof of this is worked out in detail in the classnote devoted to the epsilon - delta relationship.

5.

In[24]:=
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(* With a Do Loop *)
Do[If[Sin[n π / 4] > 0, Print[n, "  ", Sin[n π / 4]]], {n, 20}]
```

1 $\frac{1}{\sqrt{2}}$

2 1

3 $\frac{1}{\sqrt{2}}$

9 $\frac{1}{\sqrt{2}}$

10 1

11 $\frac{1}{\sqrt{2}}$

17 $\frac{1}{\sqrt{2}}$

18 1

19 $\frac{1}{\sqrt{2}}$

```
(* For statement with output omitted *)
For[n = 1, n < 21, n++, If[Sin[n π / 4] > 0, Print[n, "  ", Sin[n π / 4]]]]
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In[27]:=
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(* While statement with output omitted *)
n = 1; While[n < 21, If[Sin[n π / 4] > 0, Print[n, "  ", Sin[n π / 4]]]; n++]
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