PHYS 301
SECOND HOUR EXAM

Spring 2015

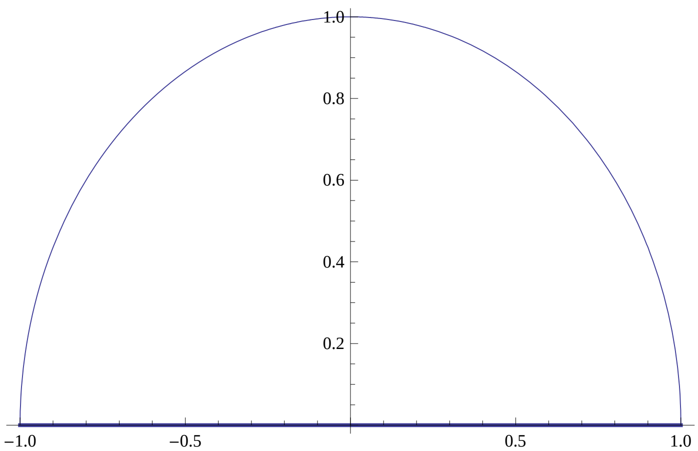
This is a closed book, closed note exam. Put all electronic devices away and out of sight now. Do all work in your blue book (s) making sure your name is on each book you use. You may do questions in any order. All solutions must show your work; you cannot receive full credit for solutions, even if they are correct, if they are not accompanied by clear and complete work. Each problem is worth 25 points.

1. Consider a force in the x - y plane :

$$\mathbf{F} = \{-y, x\}$$

Compute the work done by this force along each of these three paths (all of which lie in the x - y plane) :

a) The closed path consisting of the upper semicircle of radius 1 centered on the origin and the segment of the x axis from 1 to - 1.



b) The unit square with vertices at (0, 0) and (1, 1)

c) The square, centered on the origin, with each side of length 5.

Solution : The first step should be to compute the curl of the vector field. If the curl were zero, the field would be conservative and then you would know that the line integral of this vector around any closed path would be zero. However, the curl of this vector is found simply to be + 2 $\hat{\mathbf{z}}$. While you can compute each line integral explicitly, the most efficient method to solve this problem is to use Stokes' Theorem:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da = \int_C \mathbf{F} \cdot d\mathbf{l}$$

The normal to each area is the + z direction, so the curl integral becomes simply :

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, da \;=\; \int_S 2\, \hat{\mathbf{z}} \cdot \hat{\mathbf{z}}\, da \;=\; 2 \int da \;=\; 2\,(\text{area enclosed})$$

The line integrals for each circuit will just be twice the enclosed area, so that the line integral values for these three paths are, respectively, π, 2, and 50. You could have done these by explicit integration , but if you did, you likely ran out of time for the rest of the exam.

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2. Consider the following differential equation :

$$\left(1 - x^2\right) y'' - 3\, x\, y' + m\,(m+2)\, y = 0$$

where m is an integer. Assume a standard power series solution and determine the recursion relation for this differential equation. In the case where m = 4, write the first three non zero terms of each branch (i.e., the a_o and a_1 branches) of the solution.

Solution : We assume a power series solution of the form :

$$y \;=\; \sum_{n=0}^{\infty} a_n\, x^n \qquad y' \;=\; -3 \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \qquad y'' \;=\; \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2}$$

Substituting these expressions into the original differential equation yields :

$$\left(1 - x^2\right) \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2} \quad - 3 \sum_{n=1}^{\infty} n\, a_n\, x^n \; + m\,(m+2) \sum_{n=0}^{\infty} a_n\, x^n \;=\; 0$$

This becomes :

$$\sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2} \;-\; \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^n \;-3 \sum_{n=1}^{\infty} n\, a_n\, x^n \; + m\,(m+2) \sum_{n=0}^{\infty} a_n\, x^n \;=\; 0$$

Only the first summation needs to be re - indexed; we set k = n - 2, and using standard techniques, rewrite the series solution as :

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\, a_{n+2}\, x^n \;-\; \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^n \;-3 \sum_{n=1}^{\infty} n\, a_n\, x^n \; + m\,(m+2) \sum_{n=0}^{\infty} a_n\, x^n \;=\; 0 \qquad (1)$$

At this point, we could "strip out" terms as necessary so that all summations have the same lower limit, but I will defer that; let' s write the recursion relation directly. Grouping terms yields :

$$a_{n+2} \;=\; \frac{[n\,(n-1) + 3\,n \; - m\,(m+2)]}{(n+2)\,(n+1)}\; a_n$$

You could have stopped here to receive credit for the recursion relation, but let' s simplify the numerator on the right to help us with the next part of the problem. The numerator can be written :

$$n\,(n-1) + 3\,n - m\,(m+2) \;=\; n^2 - n + 3\,n \; - m^2 - 2\,m \;=\; \left(n^2 - m^2\right) + 2\,(n - m)$$

$$= (n+m)(n-m) + 2(n-m) = (n-m)(n+m+2)$$

Thus, the recursion relation becomes :

$$a_{n+2} = \frac{(n-m)(n+m+2)}{(n+2)(n+1)} a_n$$

For the case where m = 4, this becomes :

$$a_{n+2} = \frac{(n-4)(n+6)}{(n+2)(n+1)}$$

The coefficients for this value of m are :

$$a_2 = \frac{(0-4)(0+6)a_0}{(0+2)(0+1)} = -12\,a_0 \qquad a_4 = \frac{(2-4)(2+6)a_2}{4\cdot 3} = \frac{-16}{12}\,a_2 = +16\,a_0$$

$$a_3 = \frac{(1-4)(1+6)}{3\cdot 2}\,a_1 = \frac{-7}{2}\,a_1 \qquad a_5 = \frac{(3-4)(3+6)a_3}{5\cdot 4} = \frac{-9}{20}\,a_3 = \frac{63}{40}\,a_1$$

The solution to this differential equation are :

$$y = a_0\left(1 = 12\,x^2\ 16\,x^4\right) + a_1\left(x - \frac{7}{2}x^3 + \frac{63}{40}x^5 + \ldots\right)$$

The a_0 solution truncates after the x^4 term due to the (n-4) term in the recursion relation.

Many students incorrectly concluded that a_0, a_1 and a_2 are all zero. This cannot be the case since if it were, the solution to the differential equation would always equal zero. This incorrect conclusion derived from the attempt to “strip out” terms of the series solution. Let’s go back to eq. (1) and see what this technique yields:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - 3\sum_{n=1}^{\infty}n\,a_n x^n + m(m+2)\sum_{n=0}^{\infty}a_n x^n = 0$$

From the first and fourth sums above, we pull out the n = 0 and n = 1 terms; from the third sum we pull out the n = 1 term. Doing this will generate the following terms :

$$2a_2 + 6a_3 x - 3a_1 x + m(m+2)a_0 + m(m+2)a_1 x +$$
$$\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - 3\sum_{n=2}^{\infty}n\,a_n x^n + m(m+2)\sum_{n=2}^{\infty}a_n x^n = 0$$

Now, if we group the terms inside the summations we will obtain the same recursion relation as above. The key concept to remember here is that sum of the x^0 terms on the left equal the x^0 terms on the right (and similarly for the sum of x terms on the left and right). Grouping all the x^0 terms we get:

$$2a_2 + m(m+2)a_0 = 0 \Rightarrow a_2 = -\frac{m(m+2)}{2}a_0$$

Grouping the x terms yields :

$$6a_3 - 3a_1 x + m(m+2)a_1 x = 0 \Rightarrow a_3 = \frac{3-m(m+2)}{6}a_1$$

When m = 4, these relations tell us that $a_2 = -12\, a_0$ and $a_3 = (-7/2)\, a_1$ as we found before. This differential equation generates the Chebyshev polynomials of the second kind.

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3. The potential due to a point charge is k q/d where k is a constant, q is the charge and d is the distance between the charge and observer. Consider two charges on the x axis. A charge of + q lies at (2 a, 0), a charge of - q lies at (-a, 0). Determine the potential measured at a point O in the x - y plane that is a distance r from the origin. Assume r >> a. Express the potential in terms of appropriate sums of Legendre polynomials. What are the first two non-zero terms in the solution for the potential?

Solution :

This problem is almost identical to the dipole that we did in class and is described in detail in the classnotes. The only difference between this scenario and the classic dipole case is that the charge of + q is located at a distance of 2 a from the origin (instead of a, as it is for the dipole). We can use the law of cosines to write he potential due to the charge at 2 a is :

$$V_{+q} = \frac{k\,q}{d} = \frac{k\,q}{\sqrt{r^2 + (2\,a)^2 - 2\,(2\,a\,r)\cos\theta}} = \frac{k\,q}{r\sqrt{1 + (2\,a/r)^2 - 2\,(2\,a/r)\cos\theta}}$$

Notice how we treat the 2 a term; we express the equation in this way since we want the radical in the denominator to have the same form as the generating function for Legendre polynomials. We can see from the last term on the right that (2 a/r) has the role of h in the generating function. Following the pattern developed for the dipole case, we write the potential in terms of Legendre polynomials :

$$V = \frac{k\,q}{r}\left(\sum_{m=0}^{\infty} P_m(\cos\theta)\,(2\,a/r)^m - \sum_{m=0}^{\infty} (-1)^m\, P_m(\cos\theta)\,(a/r)^m\right)$$

In the first sum on the right, notice that the power series is now in terms of h = 2 a/r; the second term is identical to the one derived for the dipole case (since both cases involve a charge of - q a distance a from the origin). Expanding the expression for potential out to the P_3 terms yields (you were asked to include only two non zero terms):

$$V = \frac{k\,q}{r}\Big[P_0(\cos\theta)\,(2\,a/r)^0 + P_1(\cos\theta)\,(2\,a/r) + P_2(\cos\theta)\,(2\,a/r)^2 + P_3(\cos\theta)\,(2\,a/r)^3 - \left(P_0(\cos\theta)\,(a/r)^0 - P_1(\cos\theta)\,(a/r) + P_2(\cos\theta)\,(a/r)^2 - P_3(\cos\theta)\,(a/r)^3 + \ldots\right)\Big]$$

As in the case of the dipole, the P_0 terms cancel to zero, so that P_1 is the lead term, and we get:

$$V = \frac{k\,q}{r}\left[3\,P_1(\cos\theta)\,(a/r) + 3\,P_2(\cos\theta)\,(a/r)^2 + 9\,P_3(\cos\theta)\,(a/r)^3 + \ldots\right]$$

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4. An interesting dynamical system is represented by the Duffing oscillator :

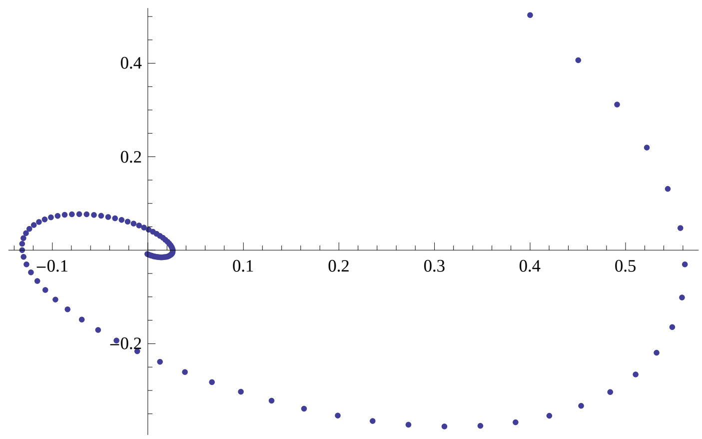
$$\ddot{x} + \delta\,\dot{x} + \alpha\,x + \beta\,x^3 = 0$$

where δ, α, β are constants. Write a Mathematica program using Euler' s method and discretization methods that will determine x (t) and v (t) for this system. Assume all constants are = 1 and the system starts at x = 0 with a velocity of 1 m/s. Your program should also plot the phase diagram (v (t) vs. x (t)) for this system. Use proper Mathematica syntax, grammar, and spelling.

Solution: I will show two solutions, one in which we define acceleration explicitly, one in which we do not. The first solution will look almost identical to the program posted in the classnote showing how to solve the harmonic oscillator with friction. In the first solution, we use the equation of motion for the Duffing oscillator and solve for acceleration (the $\ddot{x}$ term).

```
In[1]:= Clear[x, y, h, α, β, δ, a, v]
x[0] = 0; v[0] = 1; α = 1; β = 1; δ = 1; h = 0.1;
a[x_, v_] := a[x, v] = -δ v - α x - β x^3
v[n_] := v[n] = v[n - 1] + h a[x[n - 1], v[n - 1]]
x[n_] := x[n] = x[n - 1] + h v[n - 1]
ListPlot[Table[{x[n], v[n]}, {n, 100}]]
```

Out[6]=



In the second program, I do not explicitly define an acceleration term:

In[27]:=
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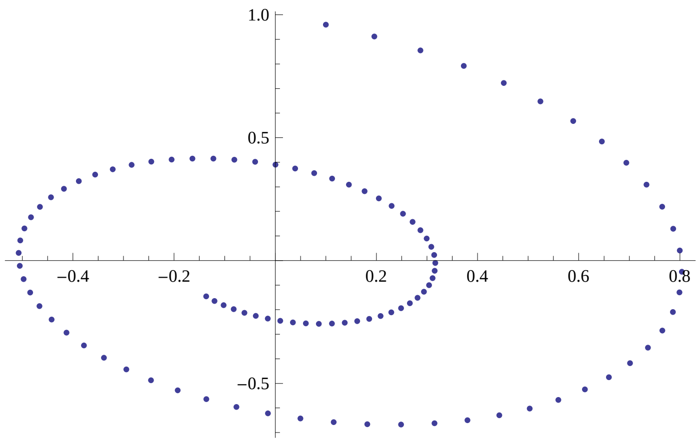
Clear[x, y, h, α, β, δ, a, v]
x[0] = 0; v[0] = 1; α = 1; β = 0.1; δ = 0.4; h = 0.1;
v[n_] := v[n] = v[n - 1] + h (-δ v[n - 1] - α x[n - 1] - β x[n - 1]^3)
x[n_] := x[n] = x[n - 1] + h v[n - 1]
ListPlot[Table[{x[n], v[n]}, {n, 100}], PlotRange → All]
```

Out[31]=



RESULTS AND FORMULAE

$$\int_V (\nabla \cdot \mathbf{v})\, d\tau = \int_S \mathbf{v} \cdot \mathbf{n}\, da$$

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, da = \int_C \mathbf{v} \cdot d\mathbf{l}$$

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$V = \sum_i V_i$$

$$g(x, h) = \frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{m=0}^{\infty} P_m(x)\, h^m$$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$f(x + h) = f(x) + h f'(x)$$

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ListPlot[Table[{Sin[n], Sin[2 n]}, {n, 50}]]
```