PHYS 301
First Hour Exam--Solutions

Spring 2016

This is a closed book, closed note exam. You will not need nor be allowed to use calculators or other electronic devices on this test. At this time, store all electronic devices, including cell phones, out of sight.

Do all your writing in your blue book (s) making sure you put your name on each blue book you use. You may do questions in any order; please make sure to label clearly the question you are solving. Your answers must show clear and complete solutions; little to no credit will be given to "correct" answers that show no work. The numbers in parentheses indicate the value of the question.

You may invoke symmetry arguments to evaluate integrals, but must explain how you are using symmetry (e.g., ‘the integrand is odd/even therefore the integral evaluated between (-L,L) is ....’.

1. Consider the function :

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

a) Determine a general expression for the Fourier coefficients (e.g., $a_n = (-1)^n/n$). You may use symmetry arguments if applicable, but be explicit in your use of them. (Hint: The example is not a correct expression for this function.) (10)

Solution : This is an odd function on the interval [-1, 1], so that 2 L = 2 and L = 1. Because the function is odd, we know the coefficients $a_0 = a_n = 0$. We find the b_n from:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin(n \pi x / L)\, dx$$

Since f is odd, the integrand is even and we can write:

$$b_n = \frac{2}{L}\int_0^L f(x) \sin(n \pi x / L)\, dx = 2\int_0^1 \sin(n \pi x)\, dx$$

$$= \frac{-2}{n\pi} \cos(n \pi x)\Big|_0^1 = \frac{-2}{n\pi}((-1)^n - 1) = \begin{cases} 4/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

b) Write the first three non - zero terms of the Fourier series. (15)

Solution : There are only sin term, so the series is represented as :

$$f(x) = \frac{4}{\pi}\left[\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \ldots\right]$$

c) Set x = 1/2 in your Fourier series. What is the value of f (1/2)? Evaluate the Fourier series at x = 1/2 and find a series representation for π. (5)

Solution : If x = 1/2, f (1/2) = 1. Set x = 1/2 in the series above :

$$f(x) = 1 = \frac{4}{\pi}\left[\sin(\pi/2) + \frac{\sin(3\pi/2)}{3} + \frac{\sin(5\pi/2)}{5} + \ldots\right]$$

$$\Rightarrow 1 = \frac{4}{\pi}[1 - 1/3 + 1/5 + \ldots] \Rightarrow \pi = 4[1 - 1/3 + 1/5 - \ldots]$$

d) Use Parseval' s Theorem to evaluate the sum : (5)

$$\sum_{n,\,\text{odd}}^{\infty} \frac{1}{n^2}$$

in other words, the sum of $1 / n^2$ over all the odd integers.

Solution : Parseval' s theorem relates the average value of f^2 to the squares of the coefficients:

$$\frac{\int_{-1}^{1} f(x)^2\, dx}{2} = a_0^2 + \frac{1}{2}\Sigma a_n^2 + \frac{1}{2}\Sigma b_n^2$$

For this function, the average value is easily shown to be 1. The b_n are the only coefficients so we have:

$$1 = \frac{1}{2} \sum_{odd}^{\infty} \left(\frac{4}{n\,\pi}\right)^2 = \frac{1}{2} \sum_{odd}^{\infty} \frac{16}{n^2\,\pi^2} \Rightarrow \sum_{odd}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

2. Each function below is $2\,\pi$ periodic. What is shown is the portion between - π and π. For each function, state whether it can be represented by a Fourier series. If the answer is yes, merely say yes. If the answer is no, explain why there is no Fourier series for that function. (Five pts for each part; a correct answer of yes earns five pts, a negative answer requires the correct reason to earn points.)

Solution : The first question examines your ability to do the mechanics of Fourier series, this question tests your understanding of the underlying theory. Dirichlet' s Conditions tells us that a function can be represented by a Fourier series if :

- It is single valued
- It has only a finite number of discontinuities and maxima/minima.
- The value of $\int_{-L}^{L} \left| f(x) \right| dx$ converges

a) $y = \frac{1}{x}$ ... No, since $\int_{-\pi}^{\pi} \left| \frac{1}{x} \right| dx = 2 \int_{0}^{\pi} \frac{1}{x} dx = 2 \ln x$ which diverges at $x = 0$

b) $y = \frac{1}{\sqrt{x}}$ ... Yes, since $\int_{-\pi}^{\pi} \left| \frac{1}{\sqrt{x}} \right| dx = 2 \int_{0}^{\pi} \frac{1}{\sqrt{x}} dx = 4 \sqrt{x} \Big|_{0}^{\pi} = 4 \sqrt{\pi}$

c) $y = \sin\left[\frac{1}{x}\right]$ ... No, since the function has an infinite number of max and min in the interval.

3. Consider the transformation equations :

$$x = \frac{1}{2}\left(u^2 - v^2\right) \qquad y = u\,v$$

a) Find the scale factors h_u and h_v (10)

Solution : We know that ds^2 must be the same in all representations, so we find dx and dy in terms of du and dv:

$$dx = u\,du - v\,dv \qquad dy = u\,dv + v\,du$$

Now, sum the squares :

$$(dx)^2 + (dy)^2 = u^2\,(du)^2 - 2\,u\,v\,du\,dv + v^2\,(dv)^2 + u^2\,(dv)^2 + 2\,u\,v\,du\,dv + v^2\,(du)^2$$
$$= \left(u^2 + v^2\right)(du)^2 + \left(u^2 + v^2\right)(dv)^2$$

so the scale factors $h_u = h_v = \sqrt{u^2 + v^2}$

b) Find the unit vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. (10)

Solution : We find the unit vectors from :

$$\hat{\mathbf{u}} = \frac{\partial \mathbf{r} / \partial \mathrm{u}}{|\partial \mathbf{r} / \partial \mathrm{u}|} \qquad \hat{\mathbf{v}} = \frac{\partial \mathbf{r} / \partial \mathrm{v}}{|\partial \mathbf{r} / \partial \mathrm{v}|}$$

where r is the position vector and:

$$\mathbf{r} = \mathrm{x}\,\hat{\mathbf{x}} + \mathrm{y}\,\hat{\mathbf{y}} + \mathrm{z}\,\hat{\mathbf{z}}$$

So,

$$\mathbf{r} = \frac{1}{2}\left(\mathrm{u}^2 - \mathrm{v}^2\right)\hat{\mathbf{x}} + \mathrm{u}\,\mathrm{v}\,\hat{\mathbf{y}}$$

$$\hat{\mathbf{u}} = \frac{\partial \mathbf{r} / \partial \mathrm{u}}{|\partial \mathbf{r} / \partial \mathrm{u}|} = \frac{\mathrm{u}\,\hat{\mathbf{x}} + \mathrm{v}\,\hat{\mathbf{y}}}{\sqrt{\mathrm{u}^2 + \mathrm{v}^2}} \qquad \hat{\mathbf{v}} = \frac{\partial \mathbf{r} / \partial \mathrm{v}}{|\partial \mathbf{r} / \partial \mathrm{v}|} = \frac{-\mathrm{v}\,\hat{\mathbf{x}} + \mathrm{u}\,\hat{\mathbf{y}}}{\sqrt{\mathrm{u}^2 + \mathrm{v}^2}}$$

c) Is the transformation orthogonal? Describe clearly your evidence for this conclusion. (5)

Solution : Yes. You have two ways to show this; the first is to note that the mixed products in part a) are zero. The second is to show that u·v = 0 and u·u = v·v = 1

4. Each δ below is a Kronecker Delta. Determine the numerical value of the product of Kronecker Deltas below : (10)

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{mp}\,\delta_{pi}$$

Solution : We contract the Kronecker deltas successively :

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{mp}\,\delta_{pi} = \delta_{ik}\,\delta_{km}\,\delta_{mp}\,\delta_{pi} = \delta_{im}\,\delta_{mp}\,\delta_{pi} = \delta_{ip}\,\delta_{pi} = \delta_{ii}$$

Now, remember that Einstein summation notation means that we sum over repeated indices :

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

5. Write the Mathematica code that will determine whether each integer between 1 and 20 is even or odd, and if the integer (n) is even, the code will print n/2. If n is odd, it will print 3 n + 1. Spelling, syntax and capitalization will count toward your score. But don' t worry about elegant output statements. (15)

```
In[4]:= Do[If[EvenQ[n], Print[n / 2], Print[3 n + 1]], {n, 20}]
```

```
4
1
10
2
16
3
22
4
28
5
34
6
40
7
46
8
52
9
58
10
```

EXTRA CREDIT: As you know, today is Leap Day. In the Gregorian calendar, we have a leap year every four years, except in century years (years ending in 00), unless the century year is evenly divisible by 400 (so 2000 was a leap year). Another leap year convention (in the revised Julian calendar) has leap years every four years, except century years, unless the century year yields a remainder of 200 or 600 when it is divided by 900 (so 2000 was a leap year since 2000/900 = 2 R 200).

What will be the first year when the Gregorian calendar and the revised Julian calendar differ? (5 pts)

The leap century years in the Gregorian calendar will be 2400, 2800, 3200. In the revised Julian calendar, 2000 and 2400 are the next leap century years. However, 2800 will not be a leap year (since 2800/900 → R100). Thus the calendars will diverge in 2800.

EQUATIONS AND RESULTS

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n \pi x/L)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n x}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i n x}\,dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n \pi x/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i n \pi x/L}$$

$$\text{average value of } (f(x))^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

$$\hat{\mathbf{q}} = \frac{\frac{\partial \mathbf{r}}{\partial q}}{\left|\frac{\partial \mathbf{r}}{\partial q}\right|}$$

(where q is a generalized spatial coordinate)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Do[*expr*, {*i*, i_{min}, i_{max}}]

starts with $i = i_{min}$

If[condition, t, f]

gives t if condition evaluates to True and f if it evaluates to False.

PrimeQ[expr]

yields True if expr is a prime number, and yields False otherwise

EvenQ[expr]

gives True if expr is an even integer, and False otherwise

OddQ[expr]

gives True if expr is an odd integer, and False otherwise

Print[expr]

prints expr as output