PHYS 301
SECOND HOUR EXAM 2016

Solutions

1. Begin with our differential equation :

$$\left(1-x^2\right) y'' - x\,y' + k^2\,y = 0$$

Our trial solution is

$$y = \sum_{n=0}^{\infty} a_n\,x^n$$

which implies

$$y' = \sum_{n=1}^{\infty} n\,a_n\,x^{n-1} \quad \text{and } y'' = \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2}$$

Substituting these into the original ODE, we get:

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n + k^2 \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Setting k = n-2 in the first sum, (and switching variables back to n) gives us:

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n - \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n + k^2 \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Combining sums :

$$\sum_{n=2}^{\infty}\left[(n+2)\,(n+1)\,a_{n+2} - \left(n\,(n-1)\,a_n - n\,a_n + k^2\right) a_n\right] x^n = 0$$

Either x = 0 (the trivia solution), or the coefficients satisfy:

$$(n+2)\,(n+1)\,a_{n+2} - \left(n\,(n-1)\,a_n - n\,a_n + k^2\right) a_n = 0$$

This yields the recursion relation:

$$a_{n+2} = \frac{n\,(n-1) + n - k^2}{(n+2)\,(n+1)}\,a_n$$

$$a_{n+2} = \frac{n^2 - k^2}{(n+2)\,(n+1)}\,a_n$$

We could strip out terms, but we would gain no information that is not already contained in the recursion relation.

b) For this part, we determine coefficients and then write the power series solution for each branch of the solution. Many students computed only the coefficients and did not write the series solution.

Note that the recursion relation suggests there is an even branch and an odd branch. Starting with the even branch :

$$n = 0 \Rightarrow a_2 = \frac{-k^2}{2} a_0$$

$$n = 2 \Rightarrow a_4 = \frac{(4-k^2)\, a_2}{4 \cdot 3} = \frac{-k^2\,(4-k\text{^}2)\, a_0}{4!}$$

The odd branch :

$$n = 1 \Rightarrow a_3 = \frac{(1-k^2)\, a_1}{3 \cdot 2}$$

$$n = 3 \Rightarrow a_5 = \frac{(9-k^2)}{5 \cdot 4} a_3 = \frac{(9-k^2)(1-k^2)\, a_1}{5!}$$

Our power series solution is:

$$y = a_0 \left(1 - \frac{k^2}{2} x^2 - \frac{k^2\,(4-k^2)}{4!} x^4 + \ldots\right) + a_1 \left(x + \frac{(1-k^2)}{3!} x^3 + \frac{(9-k^2)(1-k^2)\, x^5}{5!} + \ldots\right)$$

c) For the specific case where a_0=0 and a_1=1 and k = 7, our recursion relation tells us that the entire even branch is zero since the coefficient $a_0$is zero. If k=7, we expect that the solution will be a seventh order polynomial, and all higher order odd coefficients will be zero. We find these coefficients from the recursion relation in part a):

$$a_3 = \frac{(1^2 - 7^2)\, a_1}{3 \cdot 2} = \frac{1-49}{6} = -8$$

$$a_5 = \frac{(3^2 - 7^2)\, a_3}{5 \cdot 4} = \frac{-40}{20} a_3 = +16$$

$$a_7 = \frac{(5^2 - 7^2)\, a_5}{7 \cdot 6} = -\frac{24\,(16)}{42} = \frac{-64}{7}$$

And the complete solution is:

$$y = x - 8\, x^3 + 16\, x^5 - \frac{64}{7} x^7$$

2. For the function :

$$F = e^x \cos y\, \hat{\mathbf{x}} - e^x \sin y\, \hat{\mathbf{y}}$$

The curl is:

$$\nabla \times \mathbf{F} = \hat{\mathbf{x}} \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y\right) - \hat{\mathbf{y}} \left(\frac{\partial}{\partial x} F_z - \frac{\partial}{\partial z} F_x\right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x\right)$$

Since the function is in the x-y plane (and there is no z component), the first two terms are zero since all the partial derivatives are zero. The last term yields:

$$\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x = \frac{\partial}{\partial x}(-e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y) =$$

$$= -e^x \sin y + e^x \sin y = 0$$

This is a conservative field.

b) Find the line work along the x - axis : Many students made the mistake of assuming the work was zero since the force is conservative. The work around a closed loop is zero for a conservative force, but the work along an open segment is not necessarily zero. We can compute the integral easily along the x axis :

$$W = \int_C \mathbf{F} \cdot d\mathbf{l} = \int_C F_x\, dx + F_y\, dy$$

In this case, y = 0 so dy = 0, and the line integral simplifies to:

$$W = \int_{-3}^{3} e^x \cos y\, dx$$

The integral simplifies even further since y = 0 ⇒ cos y =1:

$$W = \int_{-3}^{3} e^x\, dx = e^3 - e^{-3}$$

c) The line integral for a conservative force is path independent. So the answer is the same as part b) since the initial and final points are the same.

3. Refer to the class notes for March 16 (proving more elaborate vector identities); this is proved in the second exam.

4. Using superposition we have for the total potential at point P :

$$V = \frac{G M}{r} + \frac{G m}{r_1} + \frac{G m}{r_2}$$

We can write the distances from the satellites to P using the law of cosines:

$$r_1 = \sqrt{r^2 + a^2 - 2\, a\, r \cos\theta}$$

It is important to remember that θ refers to the angle opposite the side whose length we are computing; in other words, it is the angle between sides a and r.

Then,

$$r_2 = \sqrt{r^2 + a^2 - 2\, a\, r \cos(90 - \theta)} = \sqrt{r^2 + a^2 - 2\, a\, r \sin\theta}$$

Assuming r > a, we can factor out an r^2term to obtain:

$$r_1 = r\sqrt{1 + (a/r)^2 - 2\,(a/r)\cos\theta}$$

$$r_2 = r\sqrt{1 + (a/r)^2 - 2\,(a/r)\sin\theta}$$

Using the expressions for distance, we can rewrite the potential as:

$$V = \frac{G\,M}{r} + \frac{G\,m}{r}\left[\frac{1}{\sqrt{1 + (a/r)^2 - 2\,(a/r)\cos\theta}} + \frac{1}{\sqrt{1 + (a/r)^2 - 2\,(a/r)\sin\theta}}\right]$$

Writing these in terms of Legendre polynomials yields :

$$V = \frac{GM}{r} + \frac{Gm}{r}\left[\sum_{m=0}^{\infty} P_m\,(\cos)\,(a/r)^m + \sum_{m=0}^{\infty} P_m\,(\sin\theta)\,(a/r)^m\right]$$

The expressions in brackets can be combined into one summation.

5. These are first order differential equations which can be solved by the techniques developed in class.

```
Clear[a, b, d, f, x, y, h]
a = 1; b = 1; d = 1; f = 1; h = 1;
x[n_] := x[n] = x[n - 1] + h (a x[n - 1] - b x[n - 1] y[n - 1])
y[n_] := y[n] = y[n - 1] + h (d x[n - 1] y[n - 1] - f y[n - 1])
ListPlot[Table[{x[n], y[n]}, {n, 1, nterms}]]
(*Where nterms is undefined.  Note the equation is solved
 exactly like all other first order differential equations *)
```