PHYS 301
FIRST HOUR EXAM SOLUTIONS

1. For the function :

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$$

we are asked to find the Fourier coefficients, the first three terms of the series, and finally use Parseval's theorem.

a) The function is 2L periodic on [-2,2] which means that 2L = 4 and L=2. Computing coefficients:

$$a_o = \frac{1}{2L}\int_0^2 1\,dx = \frac{1}{4}\cdot 2 = \frac{1}{2}$$

or, you could have just noted that this was the average value of f on the interval.

$$a_n = \frac{1}{2}\int_0^2 \cos(n\pi x/2)\,dx = \frac{1}{2}\cdot\frac{2}{n\pi}\sin(n\pi x/2)\Big|_0^2 = \frac{1}{n\pi}(\sin(n\pi) - \sin 0) = 0$$

$$b_n = \frac{1}{2}\int_0^2 \sin(n\pi x/2)\,dx = \frac{1}{2}\cdot\frac{-2}{n\pi}\cos(n\pi x/2)\Big|_0^2 = \frac{-1}{n\pi}(\cos(n\pi) - \cos(0))$$

$$= \frac{1}{n\pi}(1-(-1)^n) = \begin{cases} 0, & \text{n even} \\ 2/n\pi, & \text{n odd} \end{cases}$$

b) The Fourier series is then:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\text{Sin}[\pi x/2] + \frac{\text{Sin}[3\pi x/2]}{3} + \frac{\text{Sin}[5\pi x/2]}{5} + \ldots\right]$$

c) Parseval' s theorem states :

$$\text{average value of } (f(x))^2 \text{ on the interval } = (a_o)^2 + \frac{1}{2}\Sigma a_n^2 + \frac{1}{2}\Sigma b_n^2$$

For this function, the average value of f^2 is:

$$\frac{1}{4}\int_{-2}^2 f(x)\,dx = \frac{1}{4}\int_0^2 1\,dx = \frac{1}{2}$$

Inserting values from above:

$$\frac{1}{2} = \left(\frac{1}{2}\right)^2 + 0 + \frac{1}{2}\sum_{\text{odd}}^{\infty}\left(\frac{2}{n\pi}\right)^2$$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{2}\left(\frac{4}{\pi^2}\right)\sum_{\text{odd}}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{\text{odd}}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$

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2. For the function :

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 3 \end{cases}$$

The length of the interval is 4, so 2 L = 4 and L = 2. The Fourier coefficients are computed from :

$$a_o = \frac{1}{4}\left[\int_{-1}^{0} (-1)\, dx + \int_0^3 1\, dx\right] = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{0} -\cos(n\pi x/2)\, dx + \int_0^3 \cos(n\pi x/2)\, dx\right]$$

$$b_n = \frac{1}{2}\left[\int_{-1}^{0} -\sin(n\pi x/2)\, dx + \int_0^3 \sin(n\pi x/2)\, dx\right]$$

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3. a) Begin with the transformation equation :

$$x = \rho\cos\phi \quad y = \rho\sin\phi$$

$$dx = \cos\phi\, d\rho - \rho\sin\phi\, d\phi$$

$$dy = \sin\phi\, d\rho + \rho\cos\phi\, d\phi$$

$$(ds)^2 = (dx)^2 + (dy)^2 \Rightarrow$$

$$\begin{aligned}(ds)^2 &= \cos^2\phi\,(d\rho)^2 - 2\rho\cos\phi\sin\phi\, d\rho\, d\phi + \rho^2\sin^2\phi\,(d\phi)^2 \\ &\quad + \sin^2\phi\,(d\rho)^2 + 2\rho\cos\phi\sin\phi\, d\rho\, d\phi + \rho^2\cos^2\phi\,(d\phi)^2 \\ &= \left[\cos^2\phi + \sin^2\phi\right](d\rho)^2 + \rho^2\left(\cos^2\phi + \sin^2\phi\right)(d\phi)^2 \\ &= (d\rho)^2 + \rho^2\,(d\phi)^2\end{aligned}$$

therefore,

$$h_\rho = 1 \quad \text{and}\ h_\phi = \rho$$

b) We begin by writing the position vector :

$$\mathbf{r} = \rho\cos\phi\,\hat{\mathbf{x}} + \rho\sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\rho}} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left|\frac{\partial \mathbf{r}}{\partial \rho}\right|} = \frac{\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = \frac{\frac{\partial \mathbf{r}}{\partial \phi}}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|} = \frac{-\rho\sin\phi\,\hat{\mathbf{x}} + \rho\cos\phi\,\hat{\mathbf{y}}}{\sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

c) Given :

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}}$$

To compute the time derivative of the unit vector in ρ, we use the definition of $\hat{\boldsymbol{\rho}}$ from part b):

$$\dot{\hat{\boldsymbol{\rho}}} = \frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d}{dt}\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) = \dot{\phi}\left(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\right) = \dot{\phi}\,\hat{\boldsymbol{\phi}}$$

where we use the definition of $\hat{\boldsymbol{\phi}}$ from part b). Therefore, we can write :

$$\mathbf{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

d) If the particle moves in a circle at constant speed, we know that ρ = constant so that $\dot{\rho}$ =0, the we have for the dot product of r·v:

$$\mathbf{r}\cdot\mathbf{v} = \rho\,\hat{\boldsymbol{\rho}}\cdot\rho\,\dot{\phi}\,\hat{\boldsymbol{\phi}} = \rho^2\,\dot{\phi}\,\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\phi}}$$

Since this is an orthogonal transformation, the unit vectors are perpendicular, and the angle between them is 90^o, a result you already know from introductory physics and calculus (the velocity (tangent line) is perpendicular to the instantaneous position vector.).

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4. The Fourier coefficients for an odd function of period 1/262 s are :

$$b_n = \frac{2}{\pi n}\left(\frac{-15}{8}\cos(n\pi/2) + 1 + \frac{7}{8}\cos(n\pi)\right)$$

$$b_1 = \frac{2}{\pi}\left(1 - \frac{7}{8}\right) = \frac{1}{4\pi}$$

$$b_2 = \frac{2}{2\pi}\left(\frac{-15}{8}(-1) + 1 + \frac{7}{8}\right) = \frac{30}{8\pi} = \frac{15}{4\pi}$$

$$b_3 = \frac{2}{3\pi}\left(1 - \frac{7}{8}\right) = \frac{1}{12\pi} = \frac{1}{3}\cdot\frac{1}{4\pi}$$

$$b_4 = \frac{2}{4\pi}\left(\frac{-15}{8} + 1 + \frac{7}{8}\right) = 0$$

For this function, 2L = 1/262 s so L = 1/524 s, and the Fourier expansion is:

$$p(t) = \frac{1}{4\pi}\left[\sin(524\pi t) + 15\sin(2\cdot 524\pi t) + \frac{1}{3}\sin(3\cdot 524\pi t) + ...\right]$$

and the pressure wave looks like:

In[219]:=
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Clear[b]
b[n_] := 2/(π n) (-15 Cos[n π/2]/8 + 1 + 7 Cos[n π]/8)
Plot[Sum[b[n] Sin[524 n π t], {n, 1, 31}], {t, -1/262, 1/262},
 Ticks → {{-1/262, -3/1024, -1/524, 0, 1/524, 1/262}, {-1, 1}}]
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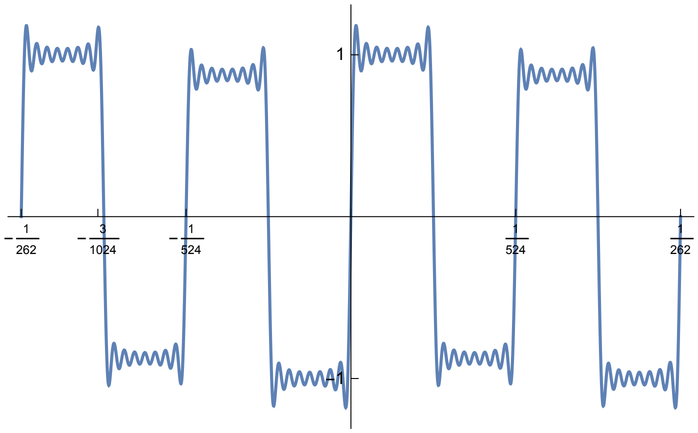
Out[221]=



Note that the coefficient of the n = 2 term is much larger than all other terms; this is the overtone that contributes the most intensity to the sound wave, and the overtone that you would hear most strongly.