PHYS 301
SECOND HOUR EXAM
SOLUTIONS

1. We begin writing our identity in summation notation :

$$\nabla \cdot (\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x_i}\epsilon_{ijk} A_j B_k = \epsilon_{ijk}\left(\frac{\partial}{\partial x_i} A_j B_k\right) = A_j \epsilon_{ijk} \frac{\partial}{\partial x_i} B_k + B_k \epsilon_{ijk} \frac{\partial}{\partial x_i} A_j$$

$$= -\mathbf{A}\cdot(\nabla\times\mathbf{B}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})$$

The first term is negative since the order of indices is anti-cyclic (i×k = - j), but the second term is cyclic and thus positive.

2. Our ODE is :

$$\left(1-x^2\right) y'' - x\,y' + \alpha^2 y = 0$$

Our trial solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting into the original ODE yields:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Re - indexing gives :

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

This yields the recursion relation:

$$a_{n+2} = n\frac{(n-1)+n - \alpha^2}{(n+2)(n+1)} a_n = \frac{n^2-\alpha^2}{(n+2)(n+1)} a_n$$

For $\alpha = 4$, we have :

$$a_2 = \frac{\left(0-4^2\right) a_0}{2\cdot} = -8\,a_o$$

$$a_4 = \frac{\left(2^2-4^2\right) a_2}{4\cdot 3} = -a_2 = +8\,a_o$$

$$a_6 = \frac{\left(4^2-4^2\right) a_4}{6\cdot 5} = 0$$

So the branch that truncates is the even branch, and its solution is:

$$y = a_o\left(1 - 8x^2 + 8x^4\right)$$

Or the Chebyshev polynomial of the fourth order.

3. This proof was worked out in detail in class.

4. Using the information detailed in classnotes and in lecture,

$$\text{we can write the total potential as : } V = k\,q\left(\frac{-1}{r_1} + \frac{1}{r_2}\right)$$

where r_1 is the distance between the -q charge and O and r_2 is the distance between q and O. If r is the distance from the origin to O, and θ is the angle between the x axis and r, we can use the law of cosines to write:

$$V = \frac{k\,q}{r}\left(\frac{1}{\sqrt{a + (a/r)^2 - 2\,(a/r)\cos(90-\theta)}} - \frac{1}{\sqrt{1 + (a/r)^2 - 2\,(a/r)\cos\theta}}\right)$$

Since $\cos(90 - \theta) = \sin\theta$, we can write these as :

$$V = \frac{k\,q}{r}\left(\sum_{m=0}^{\infty}(P_m(\sin\theta) - P_m(\cos\theta))\,(a/r)^m\right.$$

If we expand this we get:

$$V = \frac{k\,q}{r}\left[\left(P_o(\sin\theta) - P_o(\cos\theta)\,(a/r)^0 + \left(P_1(\sin\theta) - P_1(\cos\theta)\,(a/r)^1\right.\right.\right.$$

$$+\left(P_2(\sin\theta) - P_2(\cos\theta)\,(a/r)^2 + \left(P_3(\sin\theta) - P_3(\cos\theta)\,(a/r)^3 +,\,,\,,\right]\right.\right.$$

$$= \frac{k\,q}{r}\left[(1-1) + (\sin\theta - \cos\theta)\,(a/r) + \left(\frac{1}{2}\left(3\sin^2\theta - 1\right) - \frac{1}{2}\left(3\cos^2 - 1\right)\right)(a/r)^2\right.$$

$$\left. + \left(\frac{1}{2}\left(5\sin^3\theta - 3\sin\theta\right) - \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right)(a/r)^3 + 6\ldots\right]$$

more simplification can be done, but all the information is here.

5. The wave equation :

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Our trail solution is y = X(x) T(t)

Substituting into the original PDE:

$$X''\,T = \frac{1}{v^2}\,X\,T''$$

Divide by the solution:

$$\frac{X''T}{XT} = \frac{1}{v^2}\frac{XT''}{XT}$$

which leads to:

$$\frac{X''}{X} = \frac{1}{v^2}\frac{T''}{T}$$

As we described in class, we know that each side of the equation must be equal to a constant. Since the two sides are equal, they must equal the same constant. A common error made on the exam was to set one side to a positive constant and the other to a negtive constant.

Now, since we are told that the string is tied down at both ends, we can conclude that our solutions must be sinusoidal in nature, meaning that:

$$\frac{X''}{X} = -k^2 = \frac{1}{v^2}\frac{T''}{T}$$

leading to the two ODEs:

$$X'' + k^2 X = 0 \qquad T'' + k^2 v^2 T = 0$$

leading to the solution:

$$y(x, t) = (A \cos kx + B \sin kx)(C \cos(k v t) + D \sin(k v t))$$

This is as far as I asked you to go on this test. We can determine the values of A and k from the statement “the string is tied at both ends”). This tells us that y(0,t) = 0 = y(L,t)

$$y(0, t) = (A \cos 0 + B \sin 0) = 0 \Rightarrow A = 0$$

$$y(L, t) = 0 = B \sin(k L) = 0 \Rightarrow k L = n\pi \Rightarrow k = \frac{n\pi}{L}$$

We will need two statements about the string at t = 0 to determine C and D. We will examine this in class Friday.