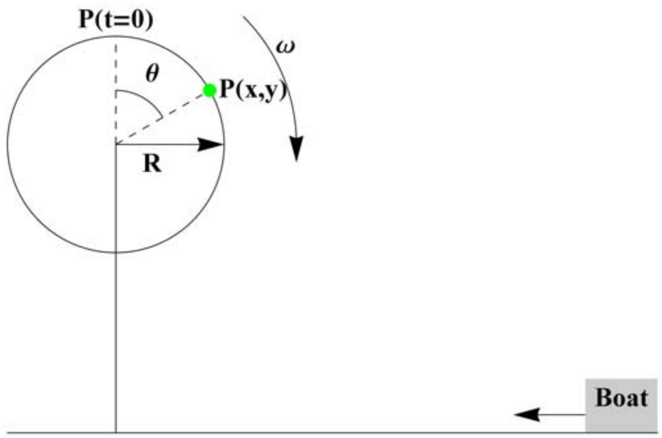
EQUATIONS OF MOTION FOR FERRIS WHEEL PROBLEM

The diagram below shows the scenario : a person starting (P(t=0)) at the top of a Ferris wheel of radius 30 m rotates at a constant angular velocity of 0.2 rad/s. Stepping off at point P(x,y), the person hits the water some time later. If the boat moves at a constant speed of 10 m/s and was initially 150 m from the base of the wheel, where will the person land with respect to the boat?



In order to compute the landing time/place of P, we need first to write the equations of motion, that is, the equations for x (t) and y (t) for P.

We adopt a coordinate system in which θ is measured clockwise from the vertical, and that the center of the Ferris wheel has Cartesian coordinates of (0, 80).

We can write the coordinates of P as :

$$\begin{aligned} P_x &= R \sin\theta \\ P_y &= H + R\cos\theta \end{aligned} \tag{1}$$

where R is the radius of the wheel (30 m) and H is the height of the center of wheel above the Lake (H = 80 m). Using vector notation, we can write the position vector of P as :

$$\mathbf{r} = R\sin\theta\,\hat{\mathbf{x}} + (H + R\cos\theta)\,\hat{\mathbf{y}} \tag{2}$$

The instantaneous velocity of P is found by taking the derivative of **r.** Remember that $\theta = \theta$ (t), so that we use the chain rule when we differentiate and obtain :

$$\mathbf{v} = \frac{d\,\mathbf{r}}{d\,t} = R\cos\theta\frac{d\theta}{d\,t}\,\hat{\mathbf{x}} - R\sin\theta\frac{d\theta}{d\,t}\,\hat{\mathbf{y}} \tag{3}$$

Recalling that d θ/d t = ω, we have :

$$\mathbf{v} = \omega R\left(\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}}\right) \tag{4}$$

This is the velocity of of the person at the time of stepping off of the wheel.

The general equations of motion are :

$$x(t) = x_o + v_{ox}\,t + \frac{1}{2}\,a_x\,t^2 \tag{5}$$

$$y(t) = y_o + v_{oy}\,t + \frac{1}{2}\,a_y\,t^2 \tag{6}$$

where x (t) and y (t) are the x, y positions at any time after leaving the wheel, the initial positions are x_o, y_o; v_{ox} and v_{oy} are the initial velocities in the x and y directions respectively, and a_x and a_y are the accelerations in the two directions. We already have expressions for the initial positions (eqs. 1) and velocity (eq. 3). Since there are no forces in the x direction, the acceleration in the x direction is zero. There is a force, gravity, in the y direction. Adopting a coordinate system in which gravity lies along the negative y axis, our equations of motion become:

$$x(t) = R\sin\theta + \omega\,R\cos\theta\,t \tag{7}$$

$$y(t) = H + R\cos\theta - \omega\,R\sin\theta\,t - \frac{1}{2}\,g\,t^2 \tag{8}$$

Now, we should remember that θ is a function of time, and we can easily express this as :

$$\theta(t) = \omega\,t \tag{9}$$

With these equations, we are ready to do some calculations. If the person starts at the top of the Ferris wheel (θ = 0, P = {0, 110}), the total time elapsed from this position until striking the water can be written as :

$$T = t_w + t_a \tag{10}$$

where T is the total time, t_w is the time on the wheel and t_a is the time in the air. The time on the wheel is simply:

$$t_w = \frac{\theta}{\omega} \tag{11}$$

When the person hits the water, y (t) = 0, so we find t_{air} by solving the quadratic equation :

$$0 = H + R\cos\theta - \omega\,R\sin\theta\,t - \frac{1}{2}\,g\,t^2 \tag{12}$$

which has solution

$$t_a = \frac{-\,\omega\,R\sin\theta \pm \sqrt{(\omega\,R\sin\theta)^2 + 2\,g\,(H + R\cos\theta)}}{g} \tag{13}$$

If we look carefully at this solution, we can see that we are interested only in the positive branch of the solution, since the negative branch would yield a negative value for time which does not make any physical sense. Therefore, we have that the total time elapsed from the beginning of the trip to hitting the water is :

$$T = t_w + t_a = \frac{\theta}{\omega} + \frac{-\omega\,R\sin\theta \pm \sqrt{(\omega\,R\sin\theta)^2 + 2\,g\,(H + R\cos\theta)}}{g} \tag{14}$$

Since the boat is traveling at a constant speed of 10 m/s and started 150 m away, the position of the boat at the moment of impact is :

$$x_{boat} = 150 - 10\,T. \tag{15}$$

Solving these equations we obtain the following output:

For 0 radians: Time on the wheel = 0. secs. Time in the air = 4.73804 secs. Impact position = 28.4282 meters. Position of the boat = 102.62 meters

For $\frac{\pi}{2}$ radians: Time on the wheel = 7.85398 secs. Time in the air = 3.47449 secs. Impact position = 30. meters. Position of the boat = 36.7153 meters

For π radians: Time on the wheel = 15.708 secs. Time in the air = 3.19438 secs. Impact position = −19.1663 meters. Position of the boat = −39.0235 meters

For $\frac{3\pi}{2}$ radians: Time on the wheel = 23.5619 secs. Time in the air = 4.69898 secs. Impact position = −30. meters. Position of the boat = −132.609 meters

Summarizing these data :

θ^o	t_w (s)	t_a (s)	x_p (m)	x_B (m)
0	0	4.74	28.4	102.6
90	7.85	3.47	30	36.71
180	15.70	3.19	−19.2	−39
270	23.56	4.69	−30	−132.6

By comparing the relative positions of the boat and person at impact, we conclude that the launch window lies somewhere in the range $90 < \theta < 180$.