EQUATIONS OF MOTION FOR FERRIS WHEEL PROBLEM

The diagram below shows the scenario: a person starting (P(t=0)) at the top of a Ferris wheel of radius 30 m rotates at a constant angular velocity of 0.2 rad/s. Stepping off at point P(x,y), the person hits the water some time later. If the boat moves at a constant speed of 10 m/s and was initially 150 m from the base of the wheel, where will the person land with respect to the boat?

In order to compute the landing time/place of P, we need first to write the equations of motion, that is, the equations for x(t) and y(t) for P.

We adopt a coordinate system in which \( \theta \) is measured clockwise from the vertical, and that the center of the Ferris wheel has Cartesian coordinates of (0, 80).

We can write the coordinates of P as:

\[
\begin{align*}
P_x &= R \sin \theta \\
P_y &= H + R \cos \theta
\end{align*}
\]

where R is the radius of the wheel (30 m) and H is the height of the center of wheel above the Lake (H = 80 m).

Using vector notation, we can write the position vector of P as:

\[
r = R \sin \theta \hat{x} + (H + R \cos \theta) \hat{y}
\]

The instantaneous velocity of P is found by taking the derivative of \( r \). Remember that \( \theta = \theta(t) \), so that we use the chain rule when we differentiate and obtain:

\[
\mathbf{v} = \frac{d \mathbf{r}}{dt} = R \cos \theta \frac{d \theta}{dt} \hat{x} - R \sin \theta \frac{d \theta}{dt} \hat{y}
\]

Recalling that \( \frac{d \theta}{dt} = \omega \), we have:

\[
\mathbf{v} = \omega R (\cos \theta \hat{x} - \sin \theta \hat{y})
\]

This is the velocity of of the person at the time of stepping off of the wheel.
The general equations of motion are:

\begin{align*}
\text{x}(t) &= x_0 + v_{ox} t + \frac{1}{2} a_x t^2 \\
\text{y}(t) &= y_0 + v_{oy} t + \frac{1}{2} a_y t^2
\end{align*}

where \(x(t)\) and \(y(t)\) are the \(x, y\) positions at any time after leaving the wheel, the initial positions are \(x_0, y_0\); \(v_{ox}\) and \(v_{oy}\) are the initial velocities in the \(x\) and \(y\) directions respectively, and \(a_x\) and \(a_y\) are the accelerations in the two directions. We already have expressions for the initial positions (eqs. 1) and velocity (eq. 3). Since there are no forces in the \(x\) direction, the acceleration in the \(x\) direction is zero. There is a force, gravity, in the \(y\) direction. Adopting a coordinate system in which gravity lies along the negative \(y\) axis, our equations of motion become:

\begin{align*}
\text{x}(t) &= R \sin \theta + \omega R \cos \theta t \\
\text{y}(t) &= H + R \cos \theta - \omega R \sin \theta t - \frac{1}{2} g t^2
\end{align*}

Now, we should remember that \(\theta\) is a function of time, and we can easily express this as:

\[ \theta(t) = \omega t \]

With these equations, we are ready to do some calculations. If the person starts at the top of the Ferris wheel (\(\theta = 0, P = \{0, 110\}\)), the total time elapsed from this position until striking the water can be written as:

\[ T = t_w + t_a \]

where \(T\) is the total time, \(t_w\) is the time on the wheel and \(t_a\) is the time in the air. The time on the wheel is simply:

\[ t_w = \frac{\theta}{\omega} \]

When the person hits the water, \(y(t) = 0\), so we find \(t_{air}\) by solving the quadratic equation:

\[ 0 = H + R \cos \theta - \omega R \sin \theta t - \frac{1}{2} g t^2 \]

which has solution

\[ t_a = \frac{-\omega R \sin \theta \pm \sqrt{(\omega R \sin \theta)^2 + 2 g (H + R \cos \theta)}}{g} \]

If we look carefully at this solution, we can see that we are interested only in the positive branch of the solution, since the negative branch would yield a negative value for time which does not make any physical sense. Therefore, we have that the total time elapsed from the beginning of the trip to hitting the water is:

\[ T = t_w + t_a = \frac{\theta}{\omega} + \frac{-\omega R \sin \theta \pm \sqrt{(\omega R \sin \theta)^2 + 2 g (H + R \cos \theta)}}{g} \]
Since the boat is traveling at a constant speed of 10 m/s and started 150 m away, the position of the boat at the moment of impact is:

\[ x_{\text{boat}} = 150 - 10 T. \] (15)

Solving these equations we obtain the following output:

For 0 radians: Time on the wheel = 0. secs. Time in the air = 4.73804 secs. Impact position = 28.4282 meters. Position of the boat = 102.62 meters

For \( \frac{\pi}{2} \) radians: Time on the wheel = 7.85398 secs. Time in the air = 3.47449 secs. Impact position = 30. meters. Position of the boat = 36.7153 meters


For \( \frac{3\pi}{2} \) radians: Time on the wheel = 23.5619 secs. Time in the air = 4.69898 secs. Impact position = -30. meters. Position of the boat = -132.609 meters

Summarizing these data:

<table>
<thead>
<tr>
<th>( \theta^0 )</th>
<th>( t_w ) (s)</th>
<th>( t_s ) (s)</th>
<th>( x_p ) (m)</th>
<th>( x_B ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4.74</td>
<td>28.4</td>
<td>102.6</td>
</tr>
<tr>
<td>90</td>
<td>7.85</td>
<td>3.47</td>
<td>30</td>
<td>36.71</td>
</tr>
<tr>
<td>180</td>
<td>15.70</td>
<td>3.19</td>
<td>-19.2</td>
<td>-39</td>
</tr>
<tr>
<td>270</td>
<td>23.56</td>
<td>4.69</td>
<td>-30</td>
<td>-132.6</td>
</tr>
</tbody>
</table>

By comparing the relative positions of the boat and person at impact, we conclude that the launch window lies somewhere in the range \( 90 < \theta < 180 \).