CONSERVATIVE FORCES AND SCALAR POTENTIALS

In our study of vector fields, we have encountered several types of conservative forces. If a force is conservative, it has a number of important properties. It is important to note that any one of the properties listed below implies all the others; in other words, if one of these properties is true for a vector field, then they are all true.

If a force is conservative :

- $\nabla \times \mathbf{F} = 0$
- the value of the line integral $\int_a^b \mathbf{F} \cdot d\mathbf{l}$ does not depend on the path, but only on the endpoints
- the line integral $\oint \mathbf{F} \cdot d\mathbf{l}$ over a closed path is zero
- $\mathbf{F}$ can be derived from a scalar potential, ϕ such that $\mathbf{F} = -\nabla \phi$

You are familiar with conservative forces; the gravitational force between point masses and the electrical force between point charges are both examples of conservative forces. This means that the work done in moving a particle from one height to another in a gravitational field depends only the mass of the object and the difference in heights; the actual path taken between the starting and end points is irrelevant.

We can easily determine whether a force is conservative by computing its curl. For instance, your last homework asked you to determine the line integral of the function :

$$\mathbf{v} = x^2 \hat{\mathbf{x}} + 2\,y\,z\,\hat{\mathbf{y}} + y^2 \hat{\mathbf{z}}$$

Over several paths from (0, 0, 0) to (1, 1, 1). If we knew a priori that the force were conservative, we would only have to calculate one line integral between these two points, knowing that the line integral for all other paths would be equal. The simplest way to determine if a force is conservative is to take its curl :

```
Needs["VectorAnalysis`"]

Curl[{x^2, 2 y z, y^2}, Cartesian[x, y, z]]

{0, 0, 0}
```

Knowing that the curl is zero, we now know that the value of $\int_{(0,0,0)}^{(1,1,1)} v \cdot dl$ will be the same for all possible paths between these limits.

We would like to be able to figure out the scalar potential that generates the vector field of the force. If we could do this, calculating line integrals becomes almost trivial, requiring only the simplest integrations. Let's review why this is the case. As we know, if **F** is conservative, then it can be derived from a scalar potential such that:

$$\mathbf{F} = -\nabla\phi = -\frac{\partial\phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial\phi}{\partial z}\hat{\mathbf{z}}$$

so that the line integral becomes :

$$W = \int_a^b \mathbf{F}\cdot d\mathbf{l} = -\int_a^b\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}}\right)\cdot(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}})$$

$$= -\int_a^b\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = -\int_a^b d\phi = -(\phi(b) - \phi(a))$$

The next to last equality occurs because the total derivative of a multivariate function f (x, y, z) is simply :

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Finally, we are left with the trivial integral of $\int_a^b d\phi$.

Let's now consider how to determine the scalar potential that generates a conservative force. If we assume that such a potential exists, then **F** = -∇ϕ, and we can equate the components of these two vectors:

$$\mathbf{F} = F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}} \text{ and } \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}}$$

Equating components, we get the three equations :

$$F_x = -\frac{\partial\phi}{\partial x};\quad F_y = -\frac{\partial\phi}{\partial y};\ F_z = -\frac{\partial\phi}{\partial z}$$

We will use the relationships to determine the scalar potential generating the function

$$F = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$$

For this given F, we know :

$F_x = x^2 = -\partial\phi/\partial x$; $F_y = 2yz = -\partial\phi/\partial y$; $F_z = y^2 = -\partial\phi/\partial z$

We are ready to begin calculating the scalar function ϕ that generates **F.** We will start with the

first relationship above:

$$-\frac{\partial \phi}{\partial x} = x^2 \Rightarrow d\phi = -\int x^2\, dx \Rightarrow \phi = \frac{-x^3}{3} + g(y, z) \tag{1}$$

It is important to understand the appearance of the g (y, z) term. In eq. (1) above, we integrate with respect to x to determine the x depence of ϕ. You learned in Calc I that indefinite integration produces a constant. In this case, g(y,z) plays that role, since g(y,z) is constant with respect to x. When we take the partial integral with respect to x, we cannot rule out the presence of any functions that depend only on y or z, so our constant of integration is written most generally as a function depending on y and z, i.e., g(y,z).

Our next step is to take the potential function as we have it in eq. (1), differentiate that with respect to y and compare this expression of $\partial\phi/\partial y$ with the expression for $\partial\phi/\partial y$ found in the equation for the force F. Taking this derivative:

$$\phi = \frac{-x^3}{3} + g(y, z) \Rightarrow \frac{\partial \phi}{\partial y} = \frac{\partial g(y, z)}{\partial y} = -2\, y\, z$$

This last equality allows us to partially integrate with respect to y and find :

$$\frac{\partial g(y, z)}{\partial y} = -2\, y\, z \Rightarrow g(y, z) = -2 \int y\, z\, dy = -y^2 z + h(z) \tag{2}$$

As before, our indefinite integral produces a constant of integration; since we integrated with respect to y, the most general way to write the constant of integration is to treat it as a function solely of z.

When we now combine eq. (1) and eq. (2), we see the scalar potential becomes :

$$\phi = \frac{-x^3}{3} - y^2 z + h(z) \tag{3}$$

Now, we take the potential function as we have it in (3), differentiate it with respect to z, and compare that result to the expression for $\partial\phi/\partial z$ in the original force equation. Taking the partial derivative of eq. (3) with respect to z yields:

$$\phi = \frac{-x^3}{3} - y^2 z + h(z) \Rightarrow \frac{\partial \phi}{\partial z} = -y^2 + \frac{dh(z)}{dz} = -y^2$$

This last differentiation shows that dh(z)/dz = 0, so that h is at most a numerical constant. We have now fully determined that the scalar potential for this force is written as:

$$\phi = \frac{-x^3}{3} - y^2 z = -\left(\frac{x^3}{3} + y^2 z\right)$$

Let' s check if this actually reproduces our force by taking the negative of the gradient of this function :

$$\mathbf{F} = -\nabla\phi = -\left(x^2\,\hat{\mathbf{x}} + 2\,y\,z\,\hat{\mathbf{y}} + y^2\,\hat{\mathbf{z}}\right)$$

and we see that the force is in fact derived from the potential in eq. (4)

Now that we know the scalar potential, we can see its utility in calculating work integrals. As shown above, the line integral for a conservative can be written as :

$$W = -\int_a^b d\phi = -(\phi(b) - \phi(a))$$

For the vector we have been using in this example (and the vector you used in homework), we can calculate the line integral simply by evaluating the scalar potential as written in eq. (5) at the end points. In other words :

$$\int_a^b d\phi = -(\phi(1, 1, 1) - \phi(0, 0, 0)) = -\left(\frac{-1}{3} - 1^2(1) - 0\right) = \frac{4}{3}$$

as you found in your homework

A Worked Example :

Let' s consider the function :

$$\mathbf{F} = 3\,x^2\,y^2\,\hat{\mathbf{x}} + \left(2\,x^3\,y + \cos z\right)\hat{\mathbf{y}} - y\sin z\,\hat{\mathbf{z}}$$

First, we will determine whether it is conservative. If it is, we will compute the scalar potential that generates it, and then calculate the line integral of this force over the path from (0, 0, 0) to $(1, 1, \pi)$

I recommend you do this by hand as an exercise :

$$\nabla\times\mathbf{F} = \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\,x^2\,y^2 & \left(2\,x^3\,y + \cos z\right) & -y\sin z \end{pmatrix} =$$

$$\hat{x}\left(-\sin z - (-\sin z) - \hat{y}\,(0 - 0) + \hat{z}\left(6\,x^2\,y - 6\,x^2\,y\right) = (0, 0, 0)\right.$$

Since the curl of F is zero, we know this is a conservative force. Now, we equate the x component of F with the x component of $\nabla\phi$, and find:

$$3\,x^2\,y^2 = -\frac{\partial\phi}{\partial x} \Rightarrow \phi = \int 3\,x^2\,y^2\,dx = -x^3\,y^2 + g(y, z)$$

Now, we differentiate the expression for ϕ immediately above with respect to y and equate to the y component of the force **F**:

$$\frac{\partial\phi}{\partial y} = -2\,x^3\,y + \frac{\partial g(y, z)}{\partial y} = -\left(2\,x^3\,y + \cos z\right)$$

This implies :

$$\frac{\partial g(y, z)}{\partial y} = \cos z \Rightarrow dg(y, z) = -\int \cos z\,dy = -y\cos z + h(z)$$

Now, we know our potential is :

$$\phi = -2\,x^3\,y - y\cos z + h(z)$$

We now differentiate this expression for ϕ with respect to z, and equate the result to the z component of the force F:

$$\frac{\partial\phi}{\partial z} = y\sin z + \frac{dh(z)}{dz} = y\sin z$$

this last relationship makes it clear h (z) is zero (or some other numerical constant), and we can write our scalar potential as :

$$\phi = -\left(2\,x^3\,y + y\cos z\right)$$

With knowledge of the potential, we can calculate the work done between any two points simply by evaluating the potential at those points. For us, the work done in exerting this force from (0, 0, 0) to (1, 1, π) is :

$$W = -(\phi(1, 1, \pi) - \phi(0, 0, 0))$$

Let' s use this opportunity to become acquainted with another interesting Mathematica operation. Look carefully at the input below which will evaluate our potential at one of the points :

```
Clear[x, y, z]
ϕ = -(2 x^3 y + y Cos[z]);
ϕ /. {x → 1, y → 1, z → π}
```

```
-1
```

After we cleared variables, we defined our scalar function. Then we made use of the "slash-dot" operation; this is a rule that says to replace x by 1, and y by 1 and z by π. Similarly, we can find

the value of ϕ at (0,0,0):

```
ϕ /. {x → 0, y → 0, z → 0}
```

```
0
```

And you obtain zero as you could easily find by inspection. Clearly, the value of the line integral is simply - 1 - 0. Since we never redefined ϕ, we can continue to use its original defintion in our succeeding "/." assignments.

Finally, we show how we could have computed the potential at the two limits in one command :

```
ϕ /. {{x → 1, y → 1, z → π}, {x → 0, y → 0, z → 0}}
```

```
{-1, 0}
```

And your results come back as in a list of output.