AN INTRODUCTION TO CURVILINEAR ORTHOGONAL COORDINATES

Overview

Throughout the first few weeks of the semester, we have studied vector calculus using almost exclusively the familiar Cartesian *x,y,z* coordinate system. In your past math and physics classes, you have encountered other coordinate systems such as cylindrical polar coordinates and spherical coordinates.

These three coordinate systems (Cartesian, cylindrical, spherical) are actually only a subset of a larger group of coordinate systems we call orthogonal coordinates. We are familiar that the unit vectors in the Cartesian system obey the relationship $x_i\, x_j\, \delta_{\mathrm{ij}}$ where δ is the Kronecker delta. In other words, the dot product of any two unit vectors is 0 unless they are the same vector (in which case the dot product is one). This is the **orthogonality** property of vectors, and orthogonal coordinate systems are those in which all the unit vectors obey this property.

We will learn general techniques for for translating vector operations into any orthogonal coordinate system, although we will be most concerned with the three systems used most frequently in basic physics (Cartesian, cylindrical polar, and spherical). We will start with very basic geometric properties of coordinate systems, and use those results as the basis for a more detailed analysis.

Scale Factors

Let' s start by considering a point in the *x-y* plane. We know we can describe the location of this point by the ordered pair (x,y) if we are using Cartesian coordinates, or by the ordered pair (ρ,ϕ) if we are using polar coordinates. You may also be familiar with the use of the symbols (r,θ) for polar coordinates; either usage is fine, but I will try to be consistent in the use of (ρ,ϕ) for plane polar coordinates, and (ρ,ϕ,z) for cylindrical polar coordinates.

For instance, the point (0,1) in Cartesian coordinates would be labeled as $(1, \pi/2)$ in polar coordinates; the Cartesian point (1,1) is equivalent to the polar coordinate position $\left(\sqrt{2}\,, \pi/4\right)$. It is a simple matter of trigonometry to show that we can transform *x,y* coordinates to ρ,ϕ coordinates via the two transformation equations:

$$\mathrm{x} = \rho \cos\phi \;\text{ and } \mathrm{y} = \rho \sin\phi \tag{1}$$

Clearly the same point will have two very different coordinate addresses when defined in different coordinate systems, but is there any property of the point' s position that is the same in both coordinate systems?

It should be clear that the distance of the point from the origin is the same no matter how we label the coordinates of the point.

We will begin our analysis of different coordinate systems with this realization that the distance between two points, no matter how you label their coordinates, must be the same in all coordinate systems.

Let' s consider two points separated by an infinitesimal distance in the x - y plane. If we call the separation between two points *s*, then then infinitesimal separation between them, ds, is given by the Pythagorean theorem as:

$$(\mathrm{ds})^2 = (\mathrm{dx})^2 + (\mathrm{dy})^2 \tag{2}$$

Now, let' s try to write this infinitesimal distance in polar coordinates. To do this, we need the transformation equations displayed in equation (1), and we also need to remember that the total derivative of a function f (u, v) is written as :

$$\mathrm{df} = \frac{\partial \mathrm{f}}{\partial \mathrm{u}}\,\mathrm{du} + \frac{\partial \mathrm{f}}{\partial \mathrm{v}}\,\mathrm{dv} \tag{3}$$

Our first step in finding the expression for $(\mathrm{ds})^2$in polar coordinates is to find the expressions for dx and dy. Using the transformation equations in (1) coupled with the definition of a total derivative in (3), we obtain:

$$\mathrm{dx} = \frac{\partial \mathrm{x}}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial \mathrm{x}}{\partial \phi}\,\mathrm{d}\phi \;\text{ and }\mathrm{dy} = \frac{\partial \mathrm{y}}{\partial \rho}\,\mathrm{d}\rho + \frac{\partial \mathrm{y}}{\partial \phi}\,\mathrm{d}\phi \tag{4}$$

Taking these total derivatives gives us :

$$\begin{aligned} \mathrm{dx} &= \cos\phi\,\mathrm{d}\rho - \rho\sin\phi\,\mathrm{d}\phi \\ \mathrm{dy} &= \sin\phi\,\mathrm{d}\rho + \rho\cos\phi\,\mathrm{d}\phi \end{aligned} \tag{5}$$

Since the distance between the two points must be the same no matter how we label them, we know that we will get the same value for $(\mathrm{ds})^2$ independent of coordinate system. Therefore, let's take the expressions for dx and dy in equations (5), and add their squares:

$$\begin{gathered} (\mathrm{ds})^2 = (\mathrm{dx})^2 + (\mathrm{dy})^2 = \\ (\cos\phi\,\mathrm{d}\rho - \rho\sin\phi\,\mathrm{d}\phi)^2 + (\sin\phi\,\mathrm{d}\rho + \rho\cos\phi\,\mathrm{d}\phi)^2 = \\ \cos^2\phi\,(\mathrm{d}\rho)^2 - 2\,\rho\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi + \rho^2\sin^2\phi\,(\mathrm{d}\phi)^2 + \\ \sin^2\phi\,(\mathrm{d}\rho)^2 + 2\,\rho\cos\phi\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi + \rho^2\cos^2\phi\,(\mathrm{d}\phi)^2 = \\ \left(\cos^2\phi + \sin^2\phi\right)(\mathrm{d}\rho)^2 + \rho^2\left(\cos^2\phi + \sin^2\phi\right)(\mathrm{d}\phi)^2 = \\ (\mathrm{d}\rho)^2 + \rho^2\,(\mathrm{d}\phi)^2 \end{gathered} \tag{6}$$

It is significant to note that all the cross terms, i.e., terms involving multiplication of $\mathrm{d}\rho$ and $\mathrm{d}\phi$ cancel out. One feature of orthogonal coordinate systems is that the expression for $(\mathrm{ds})^2$ contains no cross terms; all the terms are squares of coordinates.

Looking again at the result in equation (6), we learn that :

$$(\mathrm{dx})^2 + (\mathrm{dy})^2 = (\mathrm{d}\rho)^2 + \rho^2\,(\mathrm{d}\phi)^2 \tag{7}$$

Just as in Cartesian coordinates, the distance element $(\mathrm{ds})^2$ in polar coordinates is the sum of perfect squares. Notice that the coefficients of the differential terms (i.e., the dx and dy terms) are always one in Cartesian coordinates, but may not be in other coordinate systems.

We call the square root of these coefficients the SCALE FACTORS for the coordinate system.

For Cartesian coordinates, all scale factors are 1, so we can write :

$h_1 = h_x = 1$
$h_2 = h_y = 1$
$h_3 = h_z = 1$

Where we use h to denote scale factor. As you have seen before, we can refer to a component either by a number or a variable; the convention for Cartesian coordinates is that the variables (x, y, z) are equally represented by the components (1, 2, 3). Note also that I have included the scale factor for z even though our previous analysis was based on a two dimensional vector. For polar coordinates, our scale factors are :

$h_1 = h_\rho = 1$
$h_2 = h_\phi = \rho$
$h_3 = h_z = 1$

We haven't yet shown that $h_3 = 1$, but it is easily shown once we extend our analysis to three dimensions.

Let's think about the meaning of scale factors in different coordinate systems. Suppose we are using Cartesian coordinates to measure the position of a particles moving along the x axis. We know that if the particle moves a distance dx along the x axis, the total distance it moves, ds, is simply equal to dx. Now suppose that particle is moving along a circular arc; if we can measure that it moves through an angular displacement of $d\phi$, the total distance ds it moves is **not** merely $d\phi$; measuing in polar coordinates, the particle will move a total distance given by $ds = \rho\, d\phi$.

Let' s examine some of the ways the knowing the scale factors helps us understand the properties of a coordinate system. In the Cartesian coordinate system, we know that the incremental unit of length, **dl** is given by:

$$\mathbf{dl} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}} \tag{8}$$

Or, since all scale factors, h, have the value one in the Cartesian system, we can also write :

$$\mathbf{dl} = h_1\,dx\,\hat{\mathbf{x}} + h_2\,dy\,\hat{\mathbf{y}} + h_3\,dz\,\hat{\mathbf{z}} = h_i\,dq_i\,\hat{\mathbf{q}}_\mathbf{i} \tag{9}$$

Where the last term in (9) is the Einstein summation expression for **dl**, and where we also introduce the use of q to represent a generalized coordinate. In this usage, q_i refers to one of the coordinates in a particular system. If we are working with cylindrical coordinates, the differential element of length is then:

$$\mathbf{dl} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}}$$

Scale factors also provide us with the expressions for the differential elements of area and volume in different coordinate systems, in general :

$$dA = h_1\, h_2\, dq_1\, dq_2 \text{ and } dV = h_1\, h_2\, h_3\, dq_1\, dq_2\, dq_3$$

So in Cartesian coordinates, dA and dV are :

dA = dx dy (since the h' s are both equal to one), and dV = dx dy dz.

In cylindrical coordinates, $h_1 = 1$ and also $h_3 = 1$, but $h_\phi = \rho$, so the corresonding expressions for dA and dV become:

$$dA = \rho\, d\rho\, d\phi \;\text{ and }\; dV = \rho\, d\rho\, d\phi\, dz$$

Basis Vectors

We are very familiar with the basis vectors of the Cartesian coordinate system; these are the unit vectors that define the system, and are simply the familiar $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. We know that these are constant vectors; they have magnitude one (as do all unit vectors), and their direction in space never changes. In other words, we know that the $\hat{\boldsymbol{x}}$ vector always points in the direction of the positive x axis, and that this direction never changes in space. Because these basis vectors are constant in magnitude and direction, we have implicitly (and correctly) assumed that:

$$\frac{d}{dt}\hat{\mathbf{x}} = \frac{d}{dt}\hat{\mathbf{y}} = \frac{d}{dt}\hat{\mathbf{z}} = 0 \tag{10}$$

However, when we consider other coordinate systems, we will find that the basis vectors defining those systems have far more complex behavior. Consider Fig. 1 below. This figure shows two position vectors drawn from the origin to the edge of a circle. If we measure the location of the points P and Q in polar coordinates, we can see that the red arrows represent the direction of the unit vectors in the ρ direction, and the blue arrows represent the direction of the unit vectors in the ϕ direction.

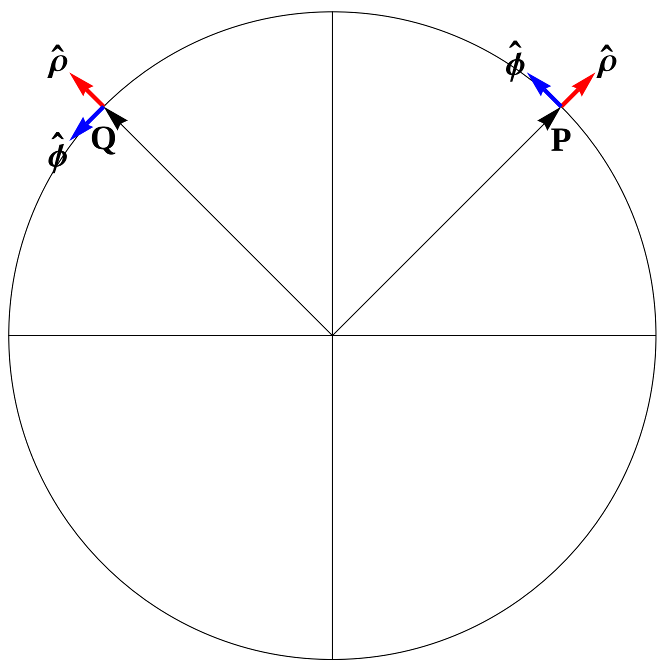

Fig. 1

It is important to recognize that while both $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ have length one, their direction in space varies depending on the position of the point on the circle. This means that when describing the motion of a particle using non-Cartesian coordinates, we must recognize that the unit vectors in non-Cartesian coordinates will not be constant in time, and that we will need to determine time derivatives of these unit vectors.

Calculating Basis Vectors

Just as the Cartesian system has the familiar basis vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, all other orthogonal coordinate systems have their own set of vectors. In polar coordinates, these vectors are $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}$; in spherical polar coordinates, they are $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$.

We are interested in expressing these basis vectors in terms of $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, and also to express $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ in terms of the basis vectors in other coordinate systems. We will be able to find these relationships by studying the properties of the position vector.

Let's start by considering the two dimensional case of plane polar coordinates. This is the system in which a particle's position is determined by knowing its distance from the origin (ρ) and the angle it makes with the positive x axis (ϕ). We know that the transformation equations between polar coordinates and Cartesian coordinates are:

$$\mathrm{x} = \rho\cos\phi \quad \text{and} \quad \mathrm{y} = \rho\sin\phi \tag{11}$$

so that we can write the position vector of a particle as :

$$\mathbf{r} = \mathrm{x}\,\hat{\mathbf{x}} + \mathrm{y}\,\hat{\mathbf{y}} \text{ or } \mathbf{r} = \rho\cos\phi\,\hat{\mathbf{x}} + \rho\sin\phi\,\hat{\mathbf{y}} \tag{12}$$

In order to find the expression for the unit vector, $\hat{\boldsymbol{\rho}}$, we compute:

$$\hat{\boldsymbol{\rho}} = \left(\frac{\partial\mathbf{r}}{\partial\rho}\right) \Big/ \left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = \frac{\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \tag{13}$$

Equation (13) relates the unit vector in the ρ direction to our familiar Cartesian basis vectors. Let' s look at (13) to see why this equation works. As we saw in equation (11), the position vector can be written as a function of both ρ and ϕ. Taking the partial derivative of **r** with respect to ρ gives us the tangent of **r** with respect to ρ; the unit vector in the ρ direction must lie in the same direction as this tangent line. Then, we divide this result by the magnitude of the vector to ensure it has length one, and we see that eq. (13) gives us the expression for the unit vector $\hat{\boldsymbol{\rho}}$**.**

Now, let's apply this same technique to determine the equation for the unit vector $\hat{\boldsymbol{\phi}}$. Start with the equation for **r,** the position vector in (12):

$$\hat{\boldsymbol{\phi}} = \left(\frac{\partial\mathbf{r}}{\partial\phi}\right) \Big/ \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{-\rho\sin\phi\,\hat{\mathbf{x}} + \rho\cos\phi\,\hat{\mathbf{y}}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \tag{14}$$

Orthogonality of Unit Vectors

The basic feature of orthogonal coordinate systems is that their basis vectors are orthogonal to each other, in other words, a coordinate system is orthogonal if and only if its unit vectors satisfy :

$$\mathrm{x_i}\,\mathrm{x_j}\,\delta_{\mathrm{ij}}$$

Do the vectors calculated in (13) and (14) satisfy these orthogonality conditions :

$$\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\rho}} = \left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right)\cdot\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) = \cos^2\phi + \sin^2\phi = 1$$

$$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \cdot (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) = \sin^2\phi + \cos^2\phi = 1$$

$$\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\phi}} = (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) \cdot (-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) = 0$$

And we have verified that these unit vectors are in fact orthogonal.

A Worked Example

Let' s consider the **parabolic cylindrical** coordinate system. The transformation equations for this coordinate system are :

$$\begin{gathered} x = \frac{1}{2}\left(u^2 - v^2\right) \\ y = u\,v \\ z = z \end{gathered} \tag{15}$$

We will follow the discussions above and compute the scale factors and basis vectors for this coordinate system, and then show that this system is in fact an orthogonal system.

In order to derive the scale factors, we first find the total derivates of dx, dy and dz :

$$\begin{gathered} dx = u\,du - v\,dv \\ dy = v\,du + u\,dv \\ dz = dz \end{gathered} \tag{16}$$

We know that the element of length must be the same no matter what coordinate systems are used; in other words :

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{17}$$

Using the expressions in (16), we square each total derivative to find :

$$\begin{gathered} (dx)^2 = u^2\,(du)^2 - 2\,uv\,du\,dv + v^2\,(dv)^2 \\ (dy)^2 = v^2\left(du^2\right) + 2\,uv\,du\,dv + u^2\,(dv)^2 \\ (dz)^2 = (dz)^2 \end{gathered} \tag{18}$$

Therefore, the square of the distance element becomes :

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \left(u^2 + v^2\right)(du)^2 + \left(u^2 + v^2\right)(dv)^2 + (dz)^2 \tag{19}$$

Notice that there are no cross product terms, indicating this is in fact an orthogonal coordinate system. The relationship in (19) allows us to assign scale factors as :

$$\begin{gathered} h_u = h_1 = \sqrt{u^2 + v^2} \\ h_v = h_2 = \sqrt{u^2 + v^2} \\ h_z = h_3 = 1 \end{gathered} \tag{20}$$

Next, we find the basis vectors $\hat{u}$, $\hat{v}$ and $\hat{z}$. It should be evident that $\hat{z} = \hat{z}$, so we will only compute the $\hat{u}$ and $\hat{v}$ unit vectors. We begin by writing the position vector:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = \frac{1}{2}\left(u^2 - v^2\right)\hat{\mathbf{x}} + u\,v\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} \tag{21}$$

Next, we take appropriate partial derivatives of the position vector and obtain :

$$\hat{\mathbf{u}} = \frac{\partial \mathbf{r}}{\partial u} \Big/ \left| \frac{\partial \mathbf{r}}{\partial u} \right| = \frac{u\,\hat{\mathbf{x}} + v\,\hat{\mathbf{y}}}{\sqrt{u^2 + v^2}}$$
$$\hat{\mathbf{v}} = \frac{\partial \mathbf{r}}{\partial v} \Big/ \left| \frac{\partial \mathbf{r}}{\partial v} \right| = \frac{-v\,\hat{\mathbf{x}} + u\,\hat{\mathbf{y}}}{\sqrt{u^2 + v^2}} \tag{22}$$

We can use the results in eq. (22) to prove that we are dealing with an orthogonal set of coordinates. To do this, we calculate the three dot products :

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \frac{u^2 + v^2}{\sqrt{u^2 + v^2}\,\sqrt{u^2 + v^2}} = \frac{u^2 + v^2}{u^2 + v^2} = 1 \tag{23}$$

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = \frac{v^2 + u^2}{\sqrt{u^2 + v^2}\,\sqrt{u^2 + v^2}} = \frac{v^2 + u^2}{u^2 + v^2} = 1$$
$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \frac{-u\,v + v\,u}{\sqrt{u^2 + v^2}\,\sqrt{u^2 + v^2}} = 0 \tag{24}$$

These dot product results prove that parabolic cyclindrical coordinates are an orthogonal system.

Question : Suppose you are presented with a coordinate system that looks similar to the parabolic cylindrical coordinate system, and has the following transformation equations:

$$x = \frac{1}{2}\left(u^2 + v^2\right) \;,\; y = u\,v \;,\; z = z$$

Is this an orthogonal coordinate system? You can certainly determine this by following all of this steps above, but can you look carefully at the transformation equations and determine this without doing any detailed calculations? Be sure you can articulate your reasoning clearly and fully.

MATHEMATICA INTERLUDE

The process we use to find scale factors is straightforward, but can involve a lot of algebraic manipulation, especially if the transformation equations become more complicated. You will find this out when you compute scale factors for the spherical coordinate system. There are some Mathematica functions you can use to ease your work, and you should feel free to use these in the upcoming homework assignment, however, if you do use them, please make sure you submit the appropriate mathematica output with your assignment.

Let' s go back to the parabolic cylindrical system, and see how Mathematica could ease our computational burden. We start with the relations expressed in eq. (16) :

$$\begin{aligned} dx &= u\,du - v\,dv \\ dy &= v\,du + u\,dv \\ dz &= dz \end{aligned}$$

We then add the squares of dx, dy and dz. The following Mathematica commands simplify this work :

```
Clear[f1, f2, f3]
f1 = u du − v dv;
f2 = v du + u dv;
f3 = dz;
Simplify[f1^2 + f2^2 + f3^2]
```

$dz^2 + (du^2 + dv^2)(u^2 + v^2)$

Notice that we first clear the variables we are about to create, and we create three new variables f1, f2 and f3, which are simply the right hand sides of the relations in (16). Notice carefully that we leave a space between u and du to indicate that u multiplies du; however, we write du without an intermediate space since we mean to define du as a single variable. The term in square brackets represents the addition of the squares of dx, dy and dz, and you see the value of the Simplify command in performing the necessary computations and presenting the output in a way that makes it very easy for you to determine the scale factors for this coordinate system.

Writing the Position Vector in Non - Cartesian Coordinates

In this section, we will see how to translate a vector from its Cartesian form and express it completely in terms of another coordinate system. Let' s start with a vector we know well, the position vector. Our goal is to transform :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} \tag{25}$$

completely into cylindrical polar coordinates. In other words, we want to write **r** as :

$$\mathbf{r} = A_\rho\,\hat{\boldsymbol{\rho}} + A_\phi\,\hat{\boldsymbol{\phi}} + A_z\,\hat{\mathbf{z}} \tag{26}$$

where the $A's$ are the coordinates of **r** in the ρ, ϕ, and z directions respectively. To make this transformation, we need to transform both the Cartesian coordinates and the Cartesian unit vectors into cylindrical coordinates. We know how to transform the coordinates by using the transformation equations for cylindrical coordinates, but we also need to transform the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ in terms of $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$. (Ok, the $\hat{z}$ transformation is trivial since the z coordinate is the same in both the Cartesian and cylindrical systems). I will leave it as an exercise for you to solve the equations in (13) and (14) of the previous write up simultaneously to show that:

$$\begin{aligned}\hat{\mathbf{x}} &= \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}\\ \hat{\mathbf{y}} &= \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}}\\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned} \tag{27}$$

These relationships and other similar ones are found on the inside back cover of Griffiths' text.

Now, substitute the transformation equations (1) along with the relations in (27) into eq. (25), and obtain:

$$\mathbf{r} = \rho\cos\phi\left(\cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}\right) + \rho\sin\phi\left(\sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\phi}\right) + z\,\hat{\mathbf{z}} \tag{28}$$

We expand terms and group them according to unit vector :

$$\mathbf{r} = \left(\rho \cos^2\phi + \rho \sin^2\phi\right)\hat{\boldsymbol{\rho}} + (-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi)\,\hat{\boldsymbol{\phi}} + z\,\hat{\mathbf{z}} \tag{29}$$

Which leads directly to :

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}} \tag{30}$$

While we have used this approach to completely translate the position vector from Cartesian to cylindrical coordinates, you can use this technique for any other vector.

Velocity, Acceleration, and Time Derivatives of Basis Vectors

Now that we have a technique for writing the position vector in any other coordinate system, we are in a position to find the first and second derivatives of **r** and derive expressions for velocity and acceleration in different coordinate systems. For our first example, we will derive the expressions for velocity and acceleration in plane polar coordinates; in other words, we will concern ourselves only with the ρ and ϕ components of velocity and acceleration, and omit any references to the z component.

Thus, in plane polar coordinates, we write the position vector simply as:

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} \tag{31}$$

and we can express velocity as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(\rho\,\hat{\boldsymbol{\rho}}\right) \tag{32}$$

Before we proceed with our differentiation, it is imperative to remember that we are not dealing with Cartesian unit vectors. Refer again to Fig. 1 above and recall that the unit vectors in non - Cartesian systems can change their directions as a particle moves. For instance, for a particle moving from point P to point Q in Fig. 1, it is easy to see that the directions of both the $\hat{\rho}$ and $\hat{\phi}$ unit vectors change direction as a function of time. Therefore, when we evaluate equation (32) above, we have to take into account that the unit vector has a non - zero time derivative. In other words, we use the product rule in eq. (32) to find :

$$\frac{d}{dt}(\rho\,\hat{\boldsymbol{\rho}}) = \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d}{dt}\,\hat{\boldsymbol{\rho}} \equiv \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\hat{\boldsymbol{\rho}}} \tag{33}$$

In equation (33), we introduce the "dot" notation. In mathematics and physics, a dot over a variable indicates differentation with respect to time, so in eq. (33) above,

$$\dot{\rho} = \frac{d\rho}{dt} \text{ and } \dot{\hat{\boldsymbol{\rho}}} = \frac{d}{dt}\,\hat{\boldsymbol{\rho}} \tag{34}$$

and a double dot over a variable means the second derivative with respect to time :

$$\ddot{\rho} = \frac{d^2\rho}{dt^2}$$

We now realize that before we can find expressions for velocity and acceleration in polar coordinates, we have to find expressions for the time derivatives of the basis vectors. To do this, we go back to equations (13) and (14) and differentiate with respect to time :

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = -\dot{\phi}\sin\phi\,\hat{\mathbf{x}} + \dot{\phi}\cos\phi\,\hat{\mathbf{y}} = \dot{\phi}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) = \dot{\phi}\,\hat{\boldsymbol{\phi}}$$

$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\dot{\phi}\cos\phi\,\hat{\mathbf{x}} - \dot{\phi}\sin\phi\,\hat{\mathbf{y}} = -\dot{\phi}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) = -\dot{\phi}\,\hat{\boldsymbol{\rho}}$$

and the equations in (35) provide us with the equations for the time derivatives of the polar basis vectors. Keep in mind that the polar angle, ϕ, may well be a function of time, so that when we differentiate sin ϕ and cos ϕ, we have to apply the chain rule; this explains the origin of the $\dot{\phi}$ terms that appear in (35).

Now, we can get back to the business of expressing velocity and acceleration in polar coordinates. Since velocity is the time derivative of the position vector, we can use the equations derived in (35) to find:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\hat{\boldsymbol{\rho}}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\phi}\,\dot{\hat{\boldsymbol{\phi}}} \tag{36}$$

Acceleration is just the time derivative of velocity, so to find acceleration in polar coordinates we take the time derivative of each term in eq. (36). However, keep in mind that the unit vectors will also vary in time, so we will wind up with five separate terms contributing to the expression for acceleration :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\rho}\,\hat{\boldsymbol{\rho}} + \dot{\rho}\,\dot{\hat{\boldsymbol{\rho}}} + \dot{\rho}\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\ddot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\dot{\phi}\,\dot{\hat{\boldsymbol{\phi}}} \tag{37}$$

We replace the $\dot{\hat{\rho}}$ and $\dot{\hat{\phi}}$ terms in (37) with the relations in (35) and get:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\rho}\,\hat{\boldsymbol{\rho}} + \dot{\rho}\left(\dot{\phi}\,\hat{\boldsymbol{\phi}}\right) + \dot{\rho}\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\ddot{\phi}\,\hat{\boldsymbol{\phi}} + \rho\,\dot{\phi}\left(-\dot{\phi}\,\hat{\boldsymbol{\rho}}\right) \tag{38}$$

Grouping terms by unit vector, we get our final expression for acceleration :

$$\mathbf{a} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{39}$$