SERIES SOLUTION TO THE DIFFERENTIAL EQUATION
DONE FOR GROUP WORK

\[ y'' + x^2 y = 0 \]

Assume a solution of the form: \[ y = \sum_{n=0}^{\infty} a_n x^n \]

Substitute the trial solution into the differential equation:

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} \]

Set \( k = n - 2 \) (so \( n = k+2 \)) in the first sum; set \( k = n+2 \) (so \( n = k -2 \)) in the second. Making these substitutions wherever there is an \( n \) yields:

\[ \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \]

Both sums have the same exponent, now we strip out the \( n = 0 \) and \( n = 1 \) terms from the first sum to get:

\[ 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-2}] x^n = 0 \]

Since all the coefficients on the left must equal their analogues on the right, we know that:

\[ a_2 = a_3 = 0 \quad \text{and} \quad a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)} \]

Because the \( (n+2) \) coefficient is related to the \( (n-2) \) coefficient, we know that \( a_6 = a_{10} (= a_{14}, \text{etc}) \) since \( a_2 = 0 \). Similarly, all the coefficients for \( n = 7, 11, 15 \) are equal to zero since \( a_3 = 0 \).

Using the recursion relation we get:

\[ a_4 = \frac{-a_0}{4 \cdot 3} \quad a_8 = \frac{-a_4}{8 \cdot 7} = \frac{+a_0}{8 \cdot 7 \cdot 4 \cdot 3} \]
\[ a_5 = \frac{-a_1}{5 \cdot 4} \quad a_9 = \frac{-a_5}{9 \cdot 8} = \frac{+a_1}{9 \cdot 8 \cdot 5 \cdot 4} \]

Using these coefficients, we get for our power series:

\[ y = a_0 \left( 1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \ldots \right) + a_1 \left( x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \ldots \right) \]

By writing the solution this way, we see there are two independent solutions, one which is odd and one which is even.