SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS— SOME WORKED EXAMPLES

First example

Let's start with a simple differential equation:

$$y'' - 2y' + y = 0 \quad (1)$$

We recognize this instantly as a second order homogeneous constant coefficient equation. Just as instantly we realize the characteristic equation has equal roots, so we can write the solution to this equation as:

$$y = e^x (A + Bx) \qquad (2)$$

where A and B are constants. Let's also assume we have the initial conditions:

$$y(0) = 1 \text{ and } y'(0) = 2$$

using these initial conditions with (2) gives us a solution to (1) of:

$$y = e^x (x + 1) \quad (3)$$

Now, let's solve this equation using series solutions methods. We do so to illustrate how this method works, and to show how the solution obtained via series methods is the same as the analytic solution, although it may not be obvious that such is the case. The fact that we used specific initial conditions will make our lives a little easier when we have to evaluate the coefficients of the series solution.

We begin our series solutions by assuming a solution to (1) of the form $y = \sum_{n=0} a_n x^n$. We will assume that all summations extend to infinity; I will omit the upper limit of the summation merely to save myself extra keystrokes in typing these symbols. We substitute this series into (1) and obtain:

$$\sum_{n=2} n(n-1)a_n x^{n-2} - 2\sum_{n=1} na_n x^{n-1} + \sum_{n=0} a_n x^n = 0 \qquad (4)$$

Our goal is to be able to equate the coefficient of each power of x that appears on the left to zero (the right side of the equation). To do so, we need to have each summation involve the same exponent of x, and also to have the same lower limit of summation.

The first step is to re-index the powers of x as they appear in each summation so that all powers of x are the same. Let's consider the first summation on the left:

$$\sum_{n=2} n(n-1)a_n x^{n-2}$$

We want the exponent of x to be n; to do this we re-index the exponent by setting $\boldsymbol{k = n-2}$. We must make this substitution everywhere n appears in the summation, including the limit, the coefficients, and the power of x. When we do this, we obtain:

$$\sum_{n=2} n(n-1)a_n x^{n-2} = \sum_{k=0} (k+2)(k+1)a_{k+2} x^k = \sum_{n=0} (n+2)(n+1)a_{n+2} x^n \qquad (5)$$

The final step is meaningful because n and k are just dummy variables, and we can call them whatever we please, so it is allowable (and typical) to write the summation in terms of the dummy index n.

Now, let's consider the summation sign in the middle term of (4). Because the exponent here is $n-1$, we re-index by setting $\boldsymbol{k = n - 1}$, and obtain:

$$\sum_{n=1} na_n x^{n-1} = \sum_{k=0} (k+1)a_{k+1} x^k = \sum_{n=0} (n+1)a_{n+1} x^n \qquad (6)$$

The final term needs no re-indexing since it is already in the form of x^n. We can take our results from (5) and (6) and substitute into (4) to write the summations as:

$$\sum_{n=0} (n+2)(n+1)a_{n+2} x^n - 2\sum_{n=0} (n+1)a_{n+1} x^n + \sum_{n=0} a_n x^n = 0 \qquad (7)$$

Since all the summations in (7) have the same limits, we can combine terms, along the way factoring out a common x^n and (7) becomes:

$$\sum x^n [(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n] = 0 \qquad (8)$$

We now reach a significant step in solving differential equations via series solutions. In order for the expression in (8) to hold for ***all*** values of x, it must be the case that the expression in brackets in (8) sums to zero for ***all*** values of n. This means that we can write:

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

Or:

$$a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+2)(n+1)} \qquad (9)$$

The relationship in (9) is called the **recursion relation** (or sometimes the recurrence relation). This tells you how to calculate the coefficients in the power series solution.

If you are not given initial or boundary conditions, you would have to calculate each of the coefficients in terms of a_0 and a_1. However, in this case we were given that $y(0) = 1$ and $y'(0) = 2$. These two conditions give us the values of a_0 and a_1 immediately. Remember that:

$$y = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n$$

Setting x=0 means all the terms in the expansion vanish except for the first term, and thus y(0) = a_0.

If we take the first derivative of y we can see that the only term that does not vanish on setting $x=0$ is the a_1 term, and so we can recognize that $y'(0) = a_1$. For the conditions given above, we have that:

$$a_0 = 1;\ a_1 = 2$$

We make use of these initial conditions, along with the recursion relation given in (9) to evaluate the coefficients that will determine our power series.

We have from (9):

$$a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+2)(n+1)} = \frac{2a_{n+1}}{n+2} - \frac{a_n}{(n+2)(n+1)}$$

For n = 0: $a_2 = \frac{2(2)}{2} - \frac{1}{2 \cdot 1} = \frac{3}{2}$

For n = 1: $a_3 = \frac{2(3/2)}{3} - \frac{2}{3 \cdot 2} = \frac{2}{3}$

For n = 2: $a_4 = \frac{2(2/3)}{4} - \frac{3/2}{4 \cdot 3} = \frac{5}{24}$

For n = 3: $a_5 = \frac{2(5/24)}{5} - \frac{2/3}{5 \cdot 4} = \frac{1}{20}$

With these coefficients, we can write the power series solution as:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots = 1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5 + \ldots \qquad (10)$$

If you substitute this into the original differential equation (1), you will find that this satisfies the equation. However, it may not be apparent that this is equivalent to the analytic solution (3). Furthermore, you might wonder where the second solution to the ODE is, since you know that a second order differential equation has two solutions.

Let's take the solution as given in (10), and rewrite it as:

$$y = 1 + (x + x) + (\frac{1}{2} + 1)x^2 + (\frac{1}{6} + \frac{1}{2})x^3 + (\frac{1}{24} + \frac{1}{6})x^4 + (\frac{1}{120} + \frac{1}{24})x^5 \quad (11)$$

This non-obvious step is taken to allow us to split (11) into two separate parts:

$$y = (1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5) + (x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5) \quad (12)$$

The first parenthetical expression is merely e^x; the second parenthetical expression allows us to factor out *x*. Upon doing this, we see immediately that the second term in (12) is simply xe^x, and our sum of solutions in (12) is identical to the analytic solution in (3).

As noted above, it is not always obvious that a series solution matches its analytic counterpart (if it has one), so it is not expected that you will be able to relate every power series solution to a simple analytic form. I will not expect you to try this; it is absolutely fine to leave solutions in the form shown in (10). However, I did want to do a more complete analysis here so you can see various aspects of how series solutions work.

Second example

This example utilizes many of the procedures we encountered in the first ODE, but adds one more important wrinkle we need to know as we solve equations via series methods.

Let's consider (this is Boas, problem 2, p. 564):

$$y' - 3x^2 y = 0 \quad (1)$$

This is a simple separable variable equation, and the solution is quickly determined to be:

$$y = A\exp(x^3) \quad (2)$$

We can also solve this via series methods by assuming a solution of the form $y = \sum_{n=0} a_n x^n$ and substituting into the original ODE. Making this substitution produces:

$$\sum_{n=1} n a_n x^{n-1} - 3\sum_{n=0} a_n x^{n+2} = 0 \quad (3)$$

Make sure you understand why x has the exponents is does in each summation. As before, we re-index the powers of x so that all powers of x equal n. In the first summation, we set ***k*=*n*-1;** in the second summation, we set ***k*=*n*+2**. With these substitutions (3) becomes:

$$\sum_{n=0}(n+1)a_{n+1}x^n - 3\sum_{n=2}a_{n-2}x^n = 0 \qquad (4)$$

We now have all the powers of x to the same value, but we are not quite ready to establish a recursion relation. This is because the summations do not have the same lower limit. In order to rewrite (4) so that both summations start at the lower limit, we make the not terribly profound but important observation that:

$$\sum_{n=0}(n+1)a_{n+1}x^n = (1)a_1x^0 + (2)a_2x^1 + \sum_{n=2}(n+1)a_{n+1}x^n \qquad (5)$$

We have "stripped out" the $n=0$ and $n=1$ terms from the first sum in eq. (4) to allow us to write that summation from the same lower limit as the second term in (4). This process of "stripping out" terms not only helps us establish the recursion relation, but also allows us to determine the values of the coefficients a_1 and a_2. Let's combine the results of (4) and (5) to write:

$$a_1 + 2a_2x + \sum_{n=2}\{(n+1)a_{n+1}x^n - 3a_{n-2}x^n\} = 0 \qquad (6)$$

One of the key aspects of series solutions is the ability to group terms on the left and equate them to zero on the right. This means that a_1 and a_2 are zero, since the x^0 and x^1 terms on the right are zero. We extend this reasoning to the expression in the summation in (6) and recognize that if this expression holds for all values of x , then it must be that:

$$(n+1)a_{n+1} - 3a_{n-2} = 0 \Rightarrow a_{n+1} = \frac{3a_{n-2}}{(n+1)} \qquad (7)$$

Eq. (7) provides us with the recursion relation for this problem. Notice that in (7), we equate the $n+1^{st}$ coefficient with the $n-2^{nd}$ coefficient; in other words, the a_0 coefficient is related to the a_3, a_6, a_9… coefficients; similarly, a_1 is related to a_4, a_7, a_{10} …, and a_2 is related to a_5, a_8, a_{11}….

Since we already know from above that $a_1 = a_2 = 0$, we know that all coefficients related to a_1 and a_2 will also be zero. We now use (7) to determine the values of those coefficients that will be non-zero:

For n =2 (remember we begin our summation at n=2): $a_3 = \frac{3a_0}{3} = a_0$

For n=5 (remember that a_{1+3n} and a_{2+3n} equal 0): $a_6 = \frac{3a_3}{6} = \frac{3a_0}{6} = \frac{a_0}{2}$

For n=8: $a_9 = \frac{3a_6}{9} = 3\frac{(a_0/2)}{9} = \frac{a_0}{6}$

For n=11: $a_{12} = \frac{3a_9}{12} = \frac{3(a_0/6)}{12} = \frac{a_0}{24}$

These coefficients allow us to write the solution to our differential equation as:

$$y = a_0(1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \frac{x^{12}}{24} + ...) = a_0 \exp(x^3) \qquad (8)$$

In this problem, we do not have initial or boundary conditions, so we have to write the solution in terms of the lead coefficient a_0. In second order equations without initial conditions, we will often find that the two solutions to the ODE involve two expressions, one involving a common factor of a_0 and the other involving a factor of a_1.

Our two examples provide us with a protocol for solving ODEs via series solutions:

1) Assume a solution of the form $y = \sum_{n=0} a_n x^n$

2) Substitute this assumed solution into the original ODE

3) Re-index summations as necessary

4) "Strip out" terms as necessary

5) Determine the recursion relation

6) Evaluate the first several non-zero coefficients

7) Use these coefficients in the series solution to determine the solution of the ODE.