Finding the Normal to a Surface

One of the elements of solving surface integrals in vector calculus is determining the normal to a surface so that we can evaluate the flux of a vector through that surface.

We can write our surface as some function :

$$f = f(x, y, z) = c \tag{1}$$

where c is a constant. For example, the equation of a plane has the form :

$$f = a x + b y + c z = d \tag{2}$$

where a, b, c, and d are constants, and the equation of the surface of a sphere is :

$$x^2 + y^2 + z^2 = a^2 \tag{3}$$

where a is the radius of the sphere. In these two cases (as in all other cases), the surface is written in the general form f (x, y, z) = c.

Now, let' s consider a point on the surface with coordinate position (x, y, z). We can write the position vector of that point with respect to the origin as :

$$\mathbf{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{4}$$

and the element of incremental length as :

$$\bar{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \tag{5}$$

Recalling the properties of derivatives, we know that **dl** must be tangent to the position vector of **r** in the surface, so that **dl** must be tangent to the surface at the point **r.**

Now, let's refer to equation (1) and take the total derivate of f:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{6}$$

and since eq. (1) tells us that f = c, we know that df = 0 so that the expression in (6) equals 0.

Let' s remember now the definition of the gradient of f :

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \tag{7}$$

We can combine equations (5) and (7) to show that df is merely the dot product of these two expressions :

$$\nabla f \cdot dl = \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \tag{8}$$

the last statement resulting because we know that df = 0. Eq. (8) tells us that ∇f·**dl** =0. This means that ∇f is normal to **dl**. Since we have already established that **dl** is tangent to the surface f(*x*, *y*, *z*), ∇f must be normal to the surface since the normal to a tangent is normal to the surface.

Let's see how we can determine the unit normal to any surface. For our first example, suppose we have the plane given as:

$$f = f(x, y, z) = 2x + 3y + 6z \tag{9}$$

First, we take the gradient to this surface :

$$\nabla f = 2\hat{x} + 3\hat{y} + 6\hat{z} \tag{10}$$

and we know that the gradient points in the direction of the normal to the surface. To find the unit normal, simply divide grad f by its length :

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{x} + 3\hat{y} + 4\hat{z}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2\hat{x} + 3\hat{y} + 4\hat{z}}{7} \tag{11}$$

In our second example, find the normal to the surface of the sphere :

$$f = f(x, y, z) = x^2 + y^2 + z^2 = a^2 \tag{12}$$

The gradient of f is :

$$\nabla f = 2x\hat{x} + 2y\hat{y} + 2z\hat{z} \tag{13}$$

And the unit normal is then :

$$\hat{\mathbf{n}} = \frac{2x\hat{x} + 2y\hat{y} + 2z\hat{z}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2x\hat{x} + 2y\hat{y} + 2z\hat{z}}{2a} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{a} = \frac{\mathbf{r}}{a} \tag{14}$$

where **r** is the position vector.