FUNCTIONAL SYMMETRY AND INTEGRALS

Overview

We have already seen that understanding the basics of symmetry of functions can help us in computing and understanding various coefficients we obtain as part of Fourier analysis.In this note, we will show how symmetry arguments can be used to simplify our work.

Odd and Even Functions

We are familiar with fundamental definitions of odd and even functions. A function is odd if f (x) = -f (-x), and a function is even if f (x) = f (-x). Not surprisingly, polynomials consisting only of odd powers of x are odd, and polynomials consisting of only even powers of x are even.

Odd and even functions also have characteristic graphs. Fig. 1 shows a plot of the odd function f (x) = $x^3 - x$ from -3 to 3:

In[116]:=
```
Plot[x^3 - x, {x, -3, 3}, Filling → Axis]
```

Out[116]=

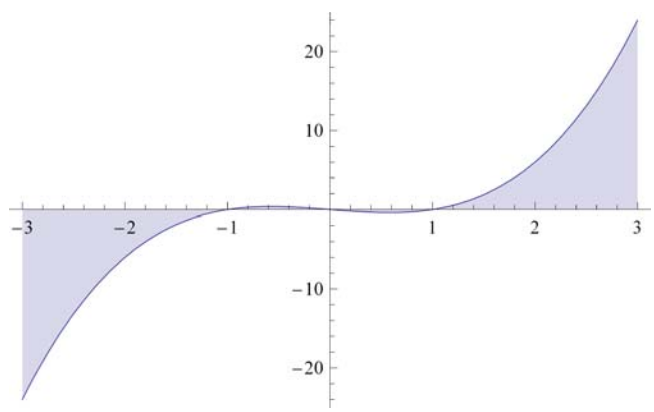


Fig. 1 Graph of $x^3 - x$ on [-3,3]

Notice also the shading between the curve and the graph. This indicates nicely that the area above the x axis is matched by the area below the x axis, so that the total integral of this function is zero between the limits [-3, 3]. This illustrates one of the key concepts of odd functions : **the integral of an odd function is zero if it is evaluated by limits that are symmetric across the origin.** We can verify this statement by explicitly calculating the integral:

In[117]:=
```
Integrate[x^3 - x, {x, -3, 3}]
```

Out[117]= 0

We can even show this symbolically; if we integrate this function between any limits that are ±a, we find:

In[118]:=
```
Integrate[x^3 - x, {x, -a, a}]
```

Out[118]= 0

Let' s consider now the properties of even functions. An even function has the basic property that f (x) = f (-x). Fig. 2 shows a plot of the function $f(x) = x^4 - x^2+3$which is an even function since each power of x is even.

In[121]:=
```
Plot[x^4 - x^2 + 3, {x, -3, 3}, Filling → Axis]
```

Out[121]=

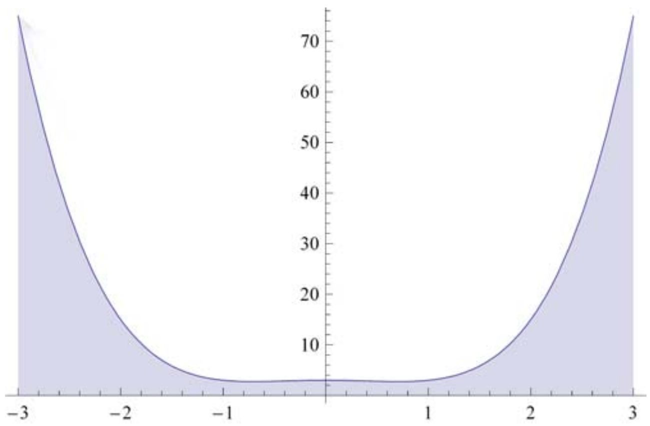


Fig. 2 Graph of $x^4 - x^2 + 3$ on [-3,3]

We can see that that the graph is symmetric with respect to the y axis, and that the area of the curve in the left half plane is equal to the area of the curve in the right half plane. In fact, even functions have the property that :

$$\int_{-a}^{a} f_{\text{even}}(x)\,dx = 2\int_{0}^{a} f_{\text{even}}(x)\,dx \qquad (1)$$

We can verify this property for this function by calculating these two integrals and taking their ratio :

In[123]:=
```
(Integrate[x^4 - x^2 + 3, {x, -a, a}]) / (2 Integrate[x^4 - x^2 + 3, {x, 0, a}])
```

Out[123]=
$$\frac{6\,a - \frac{2\,a^3}{3} + \frac{2\,a^5}{5}}{2\left(3\,a - \frac{a^3}{3} + \frac{a^5}{5}\right)}$$

In[124]:=
```
Simplify[%]
```

Out[124]= 1

The final step showing that the ratio of the two integrals is one, proving their equality.

Odd and Even Trig functions

We can extend our definitions of odd and even functions to the familiar trig functions, sine and cosine. We can consider simple plots of each function to determine their behavior :

In[127]:= `Plot[Sin[x], {x, -π, π}, Filling → Axis]`

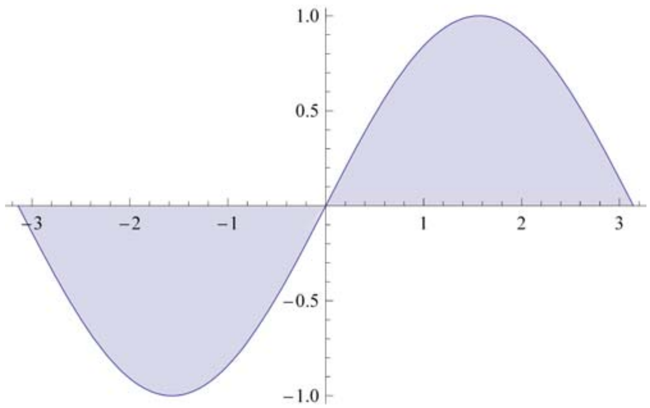
Out[127]=


Fig. 3 Graph of Sin (x)

And similarly :

In[128]:= `Plot[Cos[x], {x, -π, π}, Filling → Axis]`

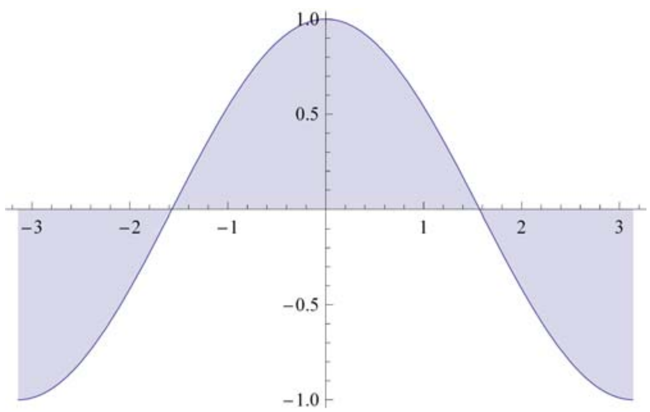
Out[128]=


Fig. 4 Graph of Cos (x)

Not only do the graphs of sin and cos reveal their symmetry, but we can also rely on the Taylor series expansions of these functions to determine their symmetry properties :

$$\begin{aligned} \operatorname{Sin} x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ \operatorname{Cos} x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \end{aligned} \tag{2}$$

Clearly, sin is an odd function since its series expansion involves only odd powers of x, and cos is an even function since its series expansion involves only even powers of x.

Products of even and odd functions

We can draw some helpful conclusions by considering products of even and odd functions. Consider the following possibilities :

• even function x even function -> even function
egs : $x^2 \cdot x^4 = x^6$; $x^2 \cdot \cos(x)$

• even function x odd function -> odd function
egs: $x^2 \cdot x = x^3$; $\cos x \cdot \sin x = \frac{1}{2}\sin(2x)$(and remember sin is an odd function)

• odd function x odd function -> even function
egs: $x \cdot x^3 = x^4$; $\sin x \cdot \sin x = \sin^2 x = 1 - \cos^2 x$ (1 and $\cos^2$ are even)

Thus, we can use combine this information with our understanding of symmetry properties of integrals to show :

• The integral of an odd function times an even function over the interval [-a, a] is zero :

```
In[129]:= Integrate[{x x^2, Cos[x] x^3, Exp[-x^2] x}, {x, -a, a}]
Out[129]= {0, 0, 0}
```

The input line above consists of a set of three products of an even function times an odd function; the output line shows each integral has a value of zero when evaluated on the interval [-a, a].

• The integral of an even function times an even function, or the integral of the product of odd functions, satisfies the relationship :

$$\int_{-a}^{a} f_1(x) f_2(x)\, dx = 2\int_{0}^{a} f_1(x) f_2(x)\, dx \qquad (3)$$

Examples :

```
In[131]:= Integrate[x^2 Cos[x], {x, -π, π}]
Out[131]= -4 π
In[132]:= Integrate[x^2 Cos[x], {x, 0, π}]
Out[132]= -2 π
In[133]:= Integrate[x Sin[x], {x, -π, π}]
Out[133]= 2 π
In[134]:= Integrate[x Sin[x], {x, 0, π}]
Out[134]= π
```