Valiantly Validating Vexing Vector Verities

1. ∇(A·B) = A×(∇×B) + B×(∇×A) + (A·∇) B + (B·∇) A

For many students, one of the most challenging vector problems is proving the identity :

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} \quad (1)$$

Many are perplexed how something so innocuous looking on the left side can generate something so complex on the right; moreover, people frequently question how can dot products and gradients, relatively friendly vector operatons, involve two expressions straight out the "BAC-CAB" horror studio?

I think part of the problem arises from years of "training" in physics and math where we look at the left side of an equation and proceed by manipulating the left side until it looks like the right. This is a case where we will start on the right and evolve an equation that proves our identity in (1)

To work with this identity, I would first start on the right and expand the terms **A×(∇×B)** and **B×(∇×A).** The game plan will be to expand those terms, construct an equation based on those expansions, and then prove the identity in (1).

Expanding **A×(∇×B)**:

$$[A\times(\nabla\times B)]_m = \epsilon_{mni}\,A_n\,\epsilon_{ijk}\,\frac{\partial}{\partial x_j}\,B_k = \epsilon_{mni}\,\epsilon_{jki}\,A_n\,\frac{\partial}{\partial x_j}\,B_k = \quad (2)$$

$$\delta_{jm}\,\delta_{kn}\,A_n\,\frac{\partial}{\partial x_j}\,B_k - \delta_{km}\,\delta_{jn}\,A_n\,\frac{\partial}{\partial x_j}\,B_k \quad (3)$$

In the first product of deltas, we realize that k = n and j = m; in the second product of deltas, k must equal m and j must equal n, so (3) becomes :

$$[A\times(\nabla\times B)]_m = A_n\,\frac{\partial}{\partial x_m}\,B_n - A_n\,\frac{\partial}{\partial x_n}\,B_m \quad (4)$$

Now, let' s expand B×(∇×A). We don't need to do another notation summation again, we merely note that B×(∇×A) is of the same form as A×(∇×B), so we just use the result from (4) and switch A and B:

$$[B\times(\nabla\times A)]_m = B_n\,\frac{\partial}{\partial x_m}\,A_n - B_n\,\frac{\partial}{\partial x_n}\,A_m \quad (5)$$

Now, I will add the results of (4) and (5):

$$[A\times(\nabla\times B)]_m + [B\times(\nabla\times A)]_m = A_n \frac{\partial}{\partial x_m} B_n - A_n \frac{\partial}{\partial x_n} B_m + B_n \frac{\partial}{\partial x_m} A_n - B_n \frac{\partial}{\partial x_n} A_m \tag{6}$$

and regroup the terms on the right as :

$$\left(A_n \frac{\partial}{\partial x_m} B_n + B_n \frac{\partial}{\partial x_m} A_n\right) - \left(A_n \frac{\partial}{\partial x_n} B_m + B_n \frac{\partial}{\partial x_n} A_m\right) \tag{7}$$

Look at the terms in the first parenthesis of (7); these two terms result from applying the product rule to:

$$\frac{\partial}{\partial x_m}(A_n B_n) = A_n \frac{\partial}{\partial x_m} B_n + B_n \frac{\partial}{\partial x_m} A_n \tag{8}$$

We recognize immediately that $A_n B_n$ is the scalar **A·B,** and $\frac{\partial}{\partial x_m}(A_n B_n)$ is just the m^{th} component of $\nabla(\boldsymbol{A}\cdot\boldsymbol{B})$ (so we finally see how $\nabla(\boldsymbol{A}\cdot\boldsymbol{B})$ has anything to do with curls and cross products.)

The terms in the second parenthesis are a little easier to decipher, the combination of $A_n \frac{\partial}{\partial x_n}$ is nothing more than the dot product between the two quantities bearing the repeated index n; $A_n \frac{\partial}{\partial x_n}$ is the summation notation way of writing $A\cdot\nabla$, so the term $A_n \frac{\partial}{\partial x_n} B_m$ is just the m^{th} component of $(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{B}$ Similarly, the last term in (7) is just the m^{th} component of **(B·∇)A.**

Now, if we sum over all m components, we can show that eq. (6) can be written:

$$\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) = \nabla(\boldsymbol{A}\cdot\boldsymbol{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{A}. \tag{9}$$

At this point, the most trivial algebraic manipulation yields :

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}. \tag{10}$$

And we have proven our identity. Give yourself a QED on the back.

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2. ∇·(A×B) = B·(∇×A) - A·(∇×B)

This is an interesting identity to study because it will give us a chance to investigate a subtlety of the permutation tensor.

Our first step, as ever, is to write the identity in summation notation. This yields :

$$\nabla\cdot(A\times B) = \frac{\partial}{\partial x_i}\epsilon_{ijk} A_j B_k \tag{11}$$

Let' s make sure we understand why (11) is written this way. The grouping $\epsilon_{ijk} A_j B_k$ represents the i^{th} term of the cross product $A\times B$. When we multiply the i^{th} component of the cross product by the i^{th} component of the operator $\frac{\partial}{\partial x_i}$, we have a product involving a repeated subscript, and we recognize that as the dot operation. In this case, the dot operation is between the differential operator and a cross product, so eq. (11) accurately expresses the identity in summation notation.

Now, since ϵ_{ijk}, A_j, and B_k are scalars, we can rearrange them as:

$$\frac{\partial}{\partial x_i}\epsilon_{ijk} A_j B_k = \epsilon_{ijk}\frac{\partial}{\partial x_i} A_j B_k \tag{12}$$

In (12) we have a differential operator acting on a product, so we apply the product rule and differentiate as :

$$\epsilon_{ijk} \frac{\partial}{\partial x_i} A_j B_k = \epsilon_{ijk}\left[B_k \frac{\partial}{\partial x_i} A_j + A_j \frac{\partial}{\partial x_i} B_k\right] \tag{13}$$

Again, exploiting the properties of working with scalars, we rearrange (13) as :

$$\epsilon_{ijk}\left[B_k \frac{\partial}{\partial x_i} A_j + A_j \frac{\partial}{\partial x_i} B_k\right] = B_k \epsilon_{ijk} \frac{\partial}{\partial x_i} A_j + A_j \epsilon_{ijk} \frac{\partial}{\partial x_i} B_k \tag{14}$$

Consider the terms in (14). The grouping $\epsilon_{ijk} \frac{\partial}{\partial x_i} A_j$ is the k^{th} component of the curl of **A.** You can see this because the ϵ symbol tells you there will be a cross product of the next term terms; since these two terms are the differential operator and A, $\epsilon_{ijk} \frac{\partial}{\partial x_i} A_j$ represents **curl A.** Since we are crossing the i^{th} component of the differential operator and the j^{th} component of **A**, we are producing the $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ component of the curl.

We can now recognize that the first term on the right in (14) is the product of $B_k (\nabla \times A)_k$ We realize immediately that this is $\boldsymbol{B} \cdot (\nabla \times \boldsymbol{A})$**.** since it is the product of a repeated index.

Now, consider the second term on the right in (14). Notice that the grouping $\epsilon_{ijk} \frac{\partial}{\partial x_i} B_k$ consists of a permutation followed by two terms; we recognize that this means the cross product of the differential operator and B, in other words, a component of **curl B**. Which component is this? Looking at the order of terms, we realize that $\epsilon_{ijk} \frac{\partial}{\partial x_i} B_k$ will produce the $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ term, so that this grouping produces the $-j$ component of **curl B.** We further recognize that we are multiplying the $-j^{th}$ component of **curl B** with the j^{th} component of A, so that the final term on the right of (14) is $-\boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$**.** And we have proven this identity.

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3. $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$

We have encountered double cross products previously in deriving the "BAC-CAB" rule, i.e.,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \tag{15}$$

We begin by setting $\mathbf{F} = \nabla \times \mathbf{A}$ and $\mathbf{G} = \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{F}$

In summation notation, we write :

$$F_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k \text{ and also } G_m = \epsilon_{mni} \frac{\partial}{\partial x_n} F_i \tag{16}$$

Since $F_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$, we have that:

$$G_m = \epsilon_{mni} \frac{\partial}{\partial x_n} (F_i) = \epsilon_{mni} \frac{\partial}{\partial x_n} \left(\epsilon_{ijk} \frac{\partial}{\partial x_j} A_k\right) = \epsilon_{mni} \epsilon_{ijk} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k \tag{17}$$

Permuting the second epsilon so that we can use the ϵ-δ relationship:

$$\epsilon_{mni} \epsilon_{ijk} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k = \epsilon_{mni} \epsilon_{jki} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k =$$

$$\left(\delta_{mj}\,\delta_{kn} - \delta_{mk}\,\delta_{jn}\right) \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k \;=\; \delta_{mj}\,\delta_{kn} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k - \delta_{mk}\,\delta_{jn} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k$$

Now, we focus our attention on the two terms at the end of eq. (18). In the first of these terms, $\delta_{mj}\,\delta_{kn} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k$, we know that j=m and also k=n else the Kronecker deltas will force the term to zero, therefore, we make the substitutions j=m and k=n and get:

$$\delta_{mj}\,\delta_{kn} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k \;=\; \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_m} A_n \tag{19}$$

Since we can always interchange order of differentiation, (19) becomes :

$$\frac{\partial}{\partial x_m} \frac{\partial}{\partial x_n} A_n = \frac{\partial}{\partial x_m} (\nabla \cdot \mathbf{A}) \;=\; \text{the } m^{th} \text{ component of } \nabla(\nabla \cdot \mathbf{A}) \tag{20}$$

We can recognize the validity of (20) by recalling that $\frac{\partial}{\partial x_n} A_n = \nabla \cdot A$, and also that $\frac{\partial}{\partial x_m} (\nabla \cdot \mathbf{A})$ represents the m^{th}component of the gradient of $\nabla \cdot \mathbf{A}$**.**

On making the substitutions k=m and j=n, the last term in (18) becomes:

$$\delta_{mk}\,\delta_{jn} \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_j} A_k \;=\; \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} A_m \tag{21}$$

Here, the repeated index, n, means to take the dot product of the two differential operators, so this term becomes :

$$\frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} A_m \;=\; (\nabla \cdot \nabla)\, A_m \tag{22}$$

or the m^{th} component of $(\nabla \cdot \nabla)\, \boldsymbol{A} \;=\; \nabla^2 \boldsymbol{A}$

If we combine our results from (20) and (22), we find that the m^{th} component of G is :

$$\mathbf{G_m} = \frac{\partial}{\partial \mathbf{x_m}} (\nabla \cdot \mathbf{A}) \;-\; (\nabla \cdot \nabla)\, \mathbf{A_m} \tag{23}$$

We can construct the complete vector **G** from this by multiplying G_m by the unit vector $\hat{e}_m$ and obtain :

$$\mathbf{G} \;=\; G_m\, \hat{\mathbf{e}}_\mathbf{m} = \frac{\partial}{\partial \mathbf{x_m}} (\nabla \cdot \mathbf{A})\, \hat{\mathbf{e}}_\mathbf{m} \;-\; (\nabla \cdot \nabla)\, \mathbf{A_m}\, \hat{\mathbf{e}}_\mathbf{m} \tag{24}$$

which becomes :

$$\mathbf{G} \;=\; \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{25}$$

QED

Remember, remember, the Kronecker Delta
And tensors permuting in turn,
With subscript location,
Summation Notation
Shall be all yours to learn.