PHYS 301
HOMEWORK #10
Solutions

1. Starting with the Legendre differential equation :

$$\left(1-x^2\right) y'' - 2\,x\,y' + m\,(m+1)\,y = 0$$

Make the substitution :

$$x = \cos\theta$$

and show the equation can be reframed as :

$$\frac{d^2 y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\,y}{d\,\theta} + m\,(m+1)\,y = 0$$

Solution : Making the substitution x = cos θ, we can transform the $\left(1-x^2\right)$ and first derivative term to obtain:

$$1-x^2 = 1-\cos^2\theta = \sin^2\theta \tag{1}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} \tag{2}$$

since x = cos θ,

$$\frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\theta}{dx} = \frac{-1}{\sin\theta} \tag{3}$$

therefore :

$$\frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta} \equiv u \tag{4}$$

Now, we wish to transform the second derivative term; using the chain rule again we have :

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{du}{dx} \tag{5}$$

$$\frac{du}{dx} = \frac{du}{d\theta}\frac{d\theta}{dx} \tag{6}$$

We already know that :

$$\frac{d\theta}{dx} = \frac{-1}{\sin\theta}$$

and $du/d\theta$ is :

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\frac{dy}{d\theta}\right) = \frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} - \frac{(-1)\,1\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} = \frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} \qquad (7)$$

Combining eqs. (3) and (7) gives :

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{d\theta}\frac{d\theta}{dx} = \left[\frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}\right]\left(\frac{-1}{\sin\theta}\right) \qquad (8)$$

Substituting eqs. (1), (4) and (8) into the original differential equation yields :

$$\left(1-x^2\right)y'' - 2\,x\,y' + m\,(m+1)\,y = \sin^2\theta\left[\frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}\right]\left(\frac{-1}{\sin\theta}\right) - 2\cos\theta\left(\frac{-1}{\sin\theta}\right)\frac{dy}{d\theta} + m\,(m+1)\,y \qquad (9)$$

A little algebra brings us to :

$$\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + 2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + m\,(m+1)\,y = \boldsymbol{\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + m\,(m+1)\,y} \qquad (10)$$

And we are done.

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2. The generating function for Bessel' s functions is :

$$g\,(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n \qquad (11)$$

where the $J_n(x)$ are the Bessel functions of the first kind of order n.

Use the generating function to show that :

a) $J_{n+1} = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ (12)

Solution : We begin by taking partial derivatives of both sides of the equation with respect to t :

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t}e^{(x/2)(t-1/t)} = (x/2)\left(1+\frac{1}{t^2}\right)e^{(x/2)(t-1/t)} = \frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} J_n(x)\,t^n = \sum_{n=-\infty}^{\infty} n\,J_n(x)\,t^{n-1} \qquad (13)$$

Now, recalling that :

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n\,, \tag{14}$$

we can rewrite the middle term in eq. (13) as :

$$(x/2)\left(1+\frac{1}{t^2}\right) e^{(x/2)(t-1/t)} =$$

$$(x/2)\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = (x/2)\left\{\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right\} \tag{15}$$

Equating the right side of eq. (15) with the right side of eq. (13) we get :

$$(x/2)\left\{\sum_{n=-\infty}^{\infty} J_n(x)\, t^n + \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-2}\right\} = \sum_{n=-\infty}^{\infty} n\, J_n(x)\, t^{n-1} \tag{16}$$

Now, we know from our previous work in series solutions that we wish to equate the coefficients of like powers of t. Now, for specificity, say we want to equate the coefficients of the t^3terms in each summation. In order to select the t^3 terms, the value of n must equal4 in the last summation on the right; n= 3 in the first summation on the left, and n must be 5 in the second summation on the left. If we generalize this result to any exponent, we see that our sums will satisfy:

$$(x/2)[J_{n-1}(x) + J_{n+1}] = n\, J_n(x) \Rightarrow \mathbf{J_{n+1} = \frac{2n}{x} J_n(x) - J_{n-1}(x)} \tag{17}$$

b) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Solution : Since we are asked to find an expression for the derivative of the Bessel function with respect to x, we might surmise that we will differentiate the generating function with respect to x :

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} e^{x/2\,(t-1/t)} = \frac{1}{2}\left(t-\frac{1}{t}\right) e^{x/2\,(t-1/t)} = \frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

Now, we recognize the exponential term is just the generating function for Bessel functions, so we can rewrite this expression as :

$$\frac{1}{2}\left(t-\frac{1}{t}\right) e^{x/2\,(t-1/t)} =$$

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n$$

Thus, if we wish to equate terms of the nth power of t, we set our coefficient equal to n in the final summation, and therefore equal to n - 1 coefficient in the first term on the left and the n + 1 coefficient for the second term on the left, giving us :

$$\frac{1}{2} J_{n-1}(x) - \frac{1}{2} J_{n+1}(x) = J_n'(x)$$

For questions 3 and 4, do all calculations by hand; you may use Mathematica to verify the results of integration, but all other work must be done by hand.

3. Expand in a Legendre series (showing the first three non zero terms) :

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

Solution : The general form of a Legendre series is :

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x)$$

$$\text{where } c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

Since our function here is odd, we can use symmetry arguments to show that all the even order coefficients will go to zero, leaving only odd coefficients. We can calculate these according to :

$$c_1 = \frac{3}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = 3 \int_0^1 1 \cdot P_1(x)\, dx = 3 \int_0^1 x\, dx = \frac{3}{2}$$

$$c_3 = 7 \int_0^1 1 \cdot \frac{1}{2} \left(5x^3 - 3x\right) dx = \frac{7}{2} \left(\frac{5}{4} - \frac{3}{2}\right) = \frac{-7}{8}$$

$$c_5 = 11 \int_0^1 1 \cdot P_5(x)\, dx = 11 \int_0^1 \frac{1}{8} \left(15x - 70x^3 + 63x^5\right) dx = \frac{11}{8} \left(\frac{15}{2} - \frac{35}{2} + \frac{21}{2}\right) = \frac{11}{16}$$

and our series is :

$$f(x) = c_1 P_1(x) + c_3 P_3(x) + c_5 P_5(x) =$$

$$\frac{3}{2} P_1 - \frac{7}{8} P_3 + \frac{11}{16} P_5 = \frac{3}{2} x - \frac{7}{8} \cdot \frac{1}{2} \left(5x^3 - 3x\right) + \frac{11}{16} \cdot \frac{1}{8} \left(15x - 70x^3 + 63x^5\right)$$

We write a short Mathematica program to plot the first 15 terms of the Legendre series and show its convergence to f (x) :

```
Clear[c, x, f, m]
f[x_] := Which[-1 < x < 0, -1, 0 < x < 1, 1]
c[m_] := ((2 m + 1 ) / 2) Integrate[f[x] LegendreP[m, x], {x, -1, 1}]
Plot[Sum[c[m] LegendreP[m, z], {m, 0, 14}], {z, -1, 1}]
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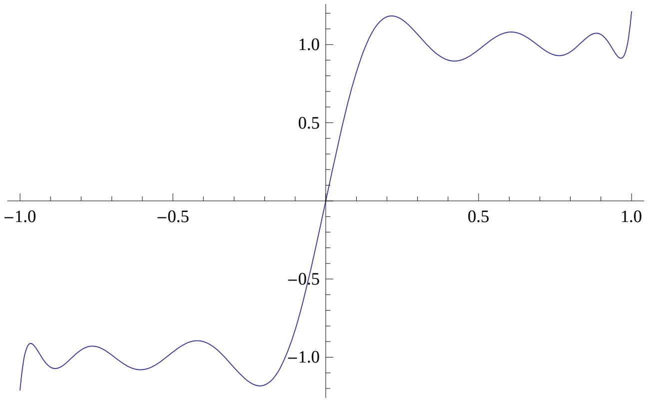

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4. Expand in a Legendre series (showing the first three non zero terms) :

$$f(x) = \arctan x \quad -1 < x < 1$$

Solution : First we note that arc tan x is an odd function (prove this to yourself either by graphing it or finding the Taylor expansion). This means that we know the Legendre series will consist only of odd terms, and we calculate the first three odd coefficients :

```
Do[ Print["c", n, " = ",
  ((2 n + 1) / 2) Integrate[LegendreP[n, x] ArcTan[x], {x, -1, 1}]], {n, 0, 5}]
```

$c0 = 0$

$c1 = \frac{3}{4}(-2 + \pi)$

$c2 = 0$

$c3 = \frac{7}{2}\left(\frac{7}{3} - \frac{3\pi}{4}\right)$

$c4 = 0$

$c5 = \frac{11}{2}\left(-\frac{106}{15} + \frac{9\pi}{4}\right)$

and along the way verify that the arc tan x is odd; with these coefficients we have :

$$f(x) = c_1 P_1(x) + c_3 P_3(x) + c_5 P_5(x)$$

We write a short Mathematica program to write the first 12 terms of the Legendre series :

```
Clear[c, f, x]
f[x_] := ArcTan[x]
c[m_] := ((2 m + 1) / 2) Integrate[f[x] LegendreP[m, x], {x, -1, 1}]

Plot[Sum[c[p] LegendreP[p, z], {p, 0, 31}], {z, -1, 1}]
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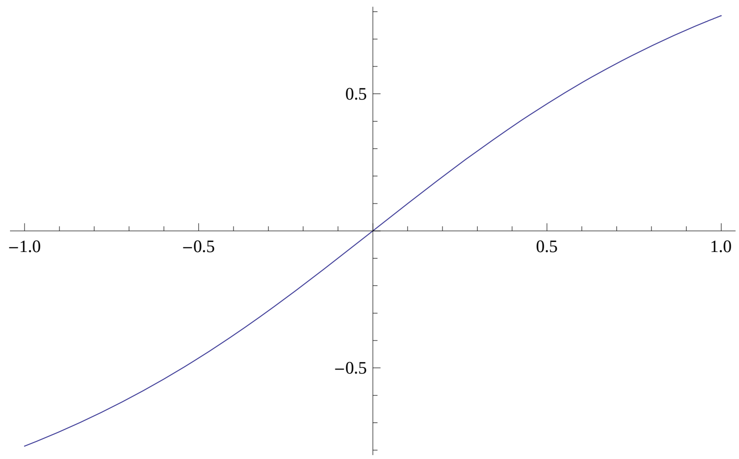
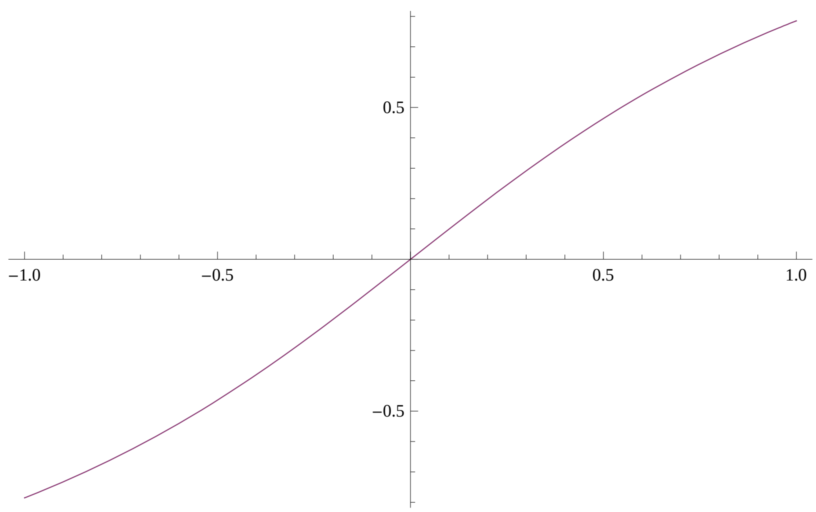


And below we superimpose the graph of the Legendre series with the graph of the arc tan function :



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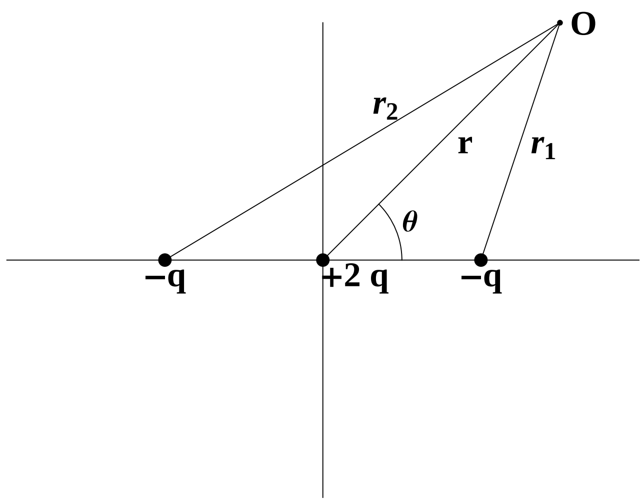
5. Consider three charges lying along the x axis. A charge of - q is at (d, 0), a charge of 2 q is at the origin, and a charge of - q lies at (-d, 0). Use Legendre polynomials to determine the potential due to this arrangement.



Solution : We find the total potential at the point O by summing the individual potentials due to the charges (the principle of superposition). The potential due to the i^{th} charge is:

$$V_i = \frac{k\,q_i}{r_i}$$

The potential due to the charge at the origin is easily expressible as :

$$V_0 = \frac{2\,k\,q}{r}$$

To express the potential due to the charge at (d,0), we use the law of cosines and write r_1 as:

$$r_1^{\,2} = r^2 + d^2 - 2\,r\,d\cos\theta \Rightarrow r_1 = r\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}$$

thus the potential due to charge 1 becomes :

$$V_1 = -\frac{k\,q}{r}\frac{1}{\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}}$$

If we look carefully at the radical, we see that this is just the generating function for Legendre Polynomials where (d/r) takes the place of h and cos θ represents x. Thus, we can write this potential as :

$$V_1 = -\frac{k\,q}{r}\frac{1}{\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}} = \sum_{m=0}^{\infty} P_m(\cos\theta)\,(d/r)^m$$

For the charge at (-d, 0), we first express the distance r_2 in terms of r, d and θ. It is important to remember here that the angle between the two lines opposite r_2 is now 180-θ, so the law of cosines gives us:

$$r_2{}^2 = r^2 + d^2 - 2\,r\,d\cos(180-\theta) = r^2 + d^2 + 2\,r\cos\theta \Rightarrow r_2 = r\sqrt{1+(d/r)^2 + 2\,r\,d\cos\theta}$$

and that the potential is expressed as :

$$V_2 = -\frac{k\,q}{r}\cdot\frac{1}{\sqrt{1+(d/r)^2+2\,(d/r)\cos\theta}}$$

Notice that the expression for V_2 differs from V_1 in the sign of the 2 (d/r) cos θ term. This is mathematically equivalent to replacing h by -h, which allows us to express V_2 as:

$$V_2 = -\frac{k\,q}{r}\cdot\frac{1}{\sqrt{1+(d/r)^2+2\,(d/r)\cos\theta}} = \sum_{m=0}^{\infty} P_m(\cos\theta)\,(-d/r)^m =$$

$$\sum_{m=0}^{\infty} (-1)^m\,P_m(\cos\theta)\,(d/r)^m$$

Thus, our total expression for the potential at point O is :

$$V = V_1 + V_2 + V_0 = \frac{-k\,q}{r}\left[\sum_{m=0}^{\infty} P_m(\cos\theta)\,(d/r)^m + \sum_{m=0}^{\infty}(-1)^m\,P_m(\cos\theta)\,(d/r)^m\right] + 2\,\frac{k\,q}{r}$$

The expression for potential involves two summations; notice that the sums add if m is even, but will cancel if m is odd, therefore we can write the total potential as :

$$V = \frac{2\,k\,q}{r}\left[1 - \sum_{m=0,\,\text{even}}^{\infty} P_m(\cos\theta)\,(d/r)^m\right]$$

Writing the first few terms explicitly gives :

$$V = \frac{2\,k\,q}{r}\left[1 - \left(1 + P_2(\cos\theta)\,(d/r)^2 + P_4(\cos\theta)\,(d/r)^4 + ...\right)\right] =$$

$$-\frac{2\,k\,q}{r}\left[\frac{1}{2}\left(3\cos^2\theta - 1\right)(d/r)^2 + \frac{1}{8}\left(3 - 30\cos^2\theta + 35\cos^4\theta\right)(d/r)^4\right]$$

If r >> d, you can see how quickly this expression converges; if we keep just the first term, we can approximate the potential field at O as :

$$V \approx -\frac{k\,q}{r}\left[\left(3\cos^2\theta - 1\right)(d/r)^2\right] = -\frac{k\,q\,d^2}{r^3}\left(3\cos^2\theta - 1\right)$$

Recalling informaton from the first part of the course, we know that the electric field is derived from the scalar potential, so that we can write :

$$\mathbf{E} = -\nabla V$$

Solving for the electric field using cylindrical coordinates:

$$-\nabla V = -\left(\frac{\partial}{\partial r} V\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta} V\,\hat{\boldsymbol{\theta}}\right) = k\,q\left(\frac{-3\,d^2}{r^4}\left(3\cos^2\theta - 1\right)\hat{\mathbf{r}} - \left(\frac{d^2}{r^4}\right)(6\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}}\right) =$$

$$-\ 3\,k\,q\left(\frac{d^2}{r^4}\right)\left[\left(3\cos^2\theta - 1\right)\hat{\mathbf{r}} + \sin 2\,\theta\,\hat{\boldsymbol{\theta}}\right]$$

If $\theta = \pi/2$, the electric field is purely radial and at large distances from the origin, and its magnitude is the well known result:

$$3\,k\,q\left(\frac{d^2}{r^4}\right)$$