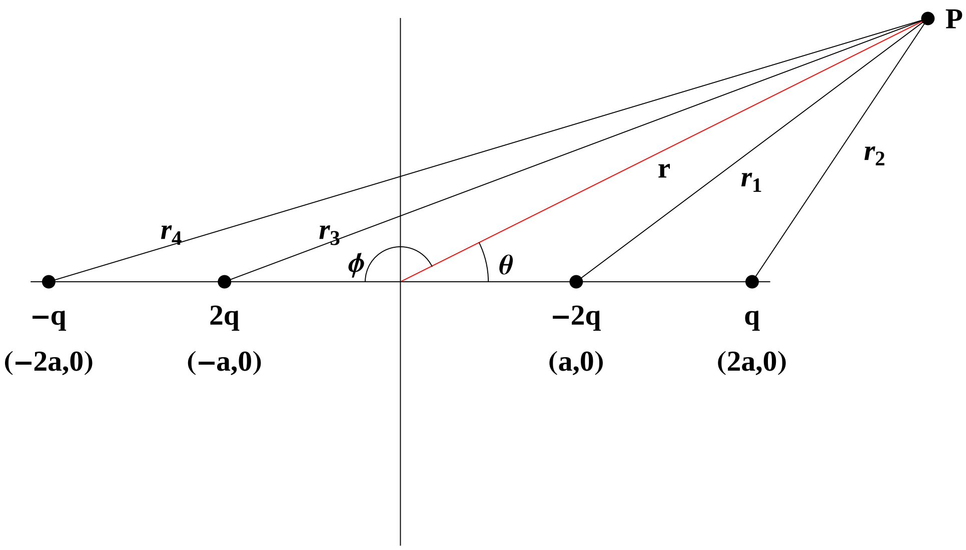
PHYS 301
HOMEWORK #11 (Optional)

Due : 27 April 2012

If you wish to have this homework set count toward your grade, please turn it in (in one electronic file or one hardcopy version) at the beginning of the last day of class, 27 April. You can pick it up from me during exam week. If you do submit for a grade, it will factor into your HW grade as would any other homework; if you do not submit it (or turn it in to be corrected but not graded) it will not affect your grade in any way. If you wish to have your homework corrected but not graded, indicate this on the first page of the assignment, otherwise I will count it toward your grade. Question 1 is worth 20 points; all other questions are worth ten points. You may but need not submit computer plots as suggested in Boas.

1. The diagram below shows a linear electric octupole:



Find the potential at point P using Legendre polynomials. Combine sums as appropriate to show that the first non - zero term involves the third order Legendre polynomial. Write out the first three non zero terms of the potential. Find the components of the electric field.

Solution : We can begin by considering these four charges as a pair of dipoles; the outer dipole consisting of the charges of magnitude q, and the inner dipole consisting of the charges whose magnitude is 2 q. Using the results previously obtained for electric dipoles, we can write the potential due to each dipole easily as :

$$V_{outer} = \frac{2\,k\,q}{r} \sum_{m=odd}^{\infty} P_m(\cos\theta)\,(2\,a/r)^m$$

$$V_{inner} = \frac{-2\,(2\,k\,q)}{r} \sum_{m=odd}^{\infty} P_m(\cos\theta)\,(a/r)^m$$

Notice that the expression for the outer potential has the term $(2\,a/r)^m$ since the distance of the charges from the origin is 2a; the inner potential term is negative since the charge closer to the observation point P is negative. Re writing the inner expression slightly we get:

$$V_{inner} = \frac{2\,k\,q}{r} \sum_{m=odd}^{\infty} 2^m\, P_m(\cos\theta)\,(a/r)^m$$

The total potential is then :

$$V_{total} = V_{outer} + V_{inner} = \frac{2\,k\,q}{r} \sum_{m=odd}^{\infty} 2^m\, P_m(\cos\theta)\,(a/r)^m - \frac{2\,(2\,k\,q)}{r} \sum_{m=odd}^{\infty} P_m(\cos\theta)\,(a/r)^m$$

$$V_{total} = \frac{2\,k\,q}{r} \sum_{m=odd}^{\infty} (2^m - 2)\, P_m(\cos\theta)\,(a/r)^m$$

When m = 1, we see that the potential equals zero; since we are summing over the odd values of m, the first non zero term is the $P_3(\cos\theta)$ term. The first three non zero terms become:

$$V \approx \frac{2\,k\,q}{r}\left(6\,P_3(\cos\theta)\,(a/r)^3 + 30\,P_5(\cos\theta)\,(a/r)^5 + 126\,P_7(\cos\theta)\,(a/r)^7 + \ldots\right)$$

So the leading term in the expansion is :

$$\frac{12\,k\,q}{r}\,P_3(\cos\theta)\,(a/r)^3 = \frac{6\,k\,q\,a^3}{r^4}\left(5\cos^3\theta - \cos\theta\right)$$

We find the electric field (for the leading term) from :

$$\mathbf{E} = -\nabla V =$$

$$-\left(\frac{\partial V}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}}\right) = +\frac{24\,k\,q\,a^3\left(5\cos^3\theta - \cos\theta\right)}{r^5}\,\hat{\mathbf{r}} - \frac{6\,k\,q\,a^3}{r^5}\left(-15\cos^2\theta\sin\theta + \sin\theta\right)\hat{\boldsymbol{\theta}}$$

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2. Show that u = f (x - v t) and u = f (x + v t) satisfy the wave equation.

Solution : The wave equation is :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Since we have that u = f (x - vt) and u = f (x + vt), we substitute these expressions into the wave equation. Starting with u = f (x - vt) :

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial x} = f'(x - vt) * 1 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = f''(x - vt)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial f}\frac{\partial f}{\partial t} = f'(x - vt)(-v) \text{ and } \frac{\partial^2 u}{\partial t^2} = f''(x - vt)(-v)^2 = v^2 f''(x - vt)$$

Using these results we get :

$$f''(x - vt) = \frac{1}{v^2}\left(v^2 f''(x - vt)\right) \Rightarrow f''(x - vt) = f''(x - vt)$$

which shows that u = f (x - vt) satisfies the wave equation. An identical analysis will show that f (x + vt) satisfies the wave equation.

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3. Boas, problem 1 page 626.

We will follow the work done in Boas and realize that our general solution will be of the form :

$$T(x, y) = \sum_{k=1}^{\infty} (A_k \cos kx + B_k \sin kx)\left(Ce^{ky} + De^{-ky}\right)$$

As we did in class, we know that applying boundary conditions gives us :

- C = 0 since the temperature must be finite for large y
- A = 0 since T = 0 whenever x = 0
- Since sin (kx) = 0 when x = 10, we have that sin (10 k) = 0 ⇒ $k = n\pi/10$

These conditions lead to the equation :

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) e^{-n\pi y/10} \tag{1}$$

Applying the final boundary condition, that T (x, 0) = x, we get :

$$T(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) = x$$

The only remaining step to getting a complete solution is evaluation of the B coefficients; we realize that the B coefficients are merely the Fourier coefficients for the series expansion of f (x) = x on -10 < x < 10. Thus we have :

$$B_n = b_n = \frac{2}{10}\int_0^{10} x \sin(n\pi x/10)\, dx$$

using Mathematica to find the coefficients :

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(2 / 10) Integrate[x Sin[n π x / 10], {x, 0, 10}]
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$$-\frac{20\,(n\,\pi\,\mathrm{Cos}[n\,\pi]-\mathrm{Sin}[n\,\pi])}{n^2\,\pi^2}$$

and we see readily that the coefficients are given by :

$$B_n = b_n = \frac{-20}{n\pi}(-1)^n = \frac{20\,(-1)^{n+1}}{n\pi}$$

using this value in equation (1) yields our final solution :

$$T(x, y) = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\left(\frac{n\pi x}{10}\right)e^{-n\pi y/10}}{n}$$

This temperature distribution yields the contour plot :

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Clear[temp]
temp[x_, y_] := (20/π) Sum[((-1)^(n+1) Sin[n π x/10] Exp[-n π y / 10])/n, {n, 1, 51}]
ContourPlot[temp[x, y], {x, 0, 10}, {y, 0, 20}]
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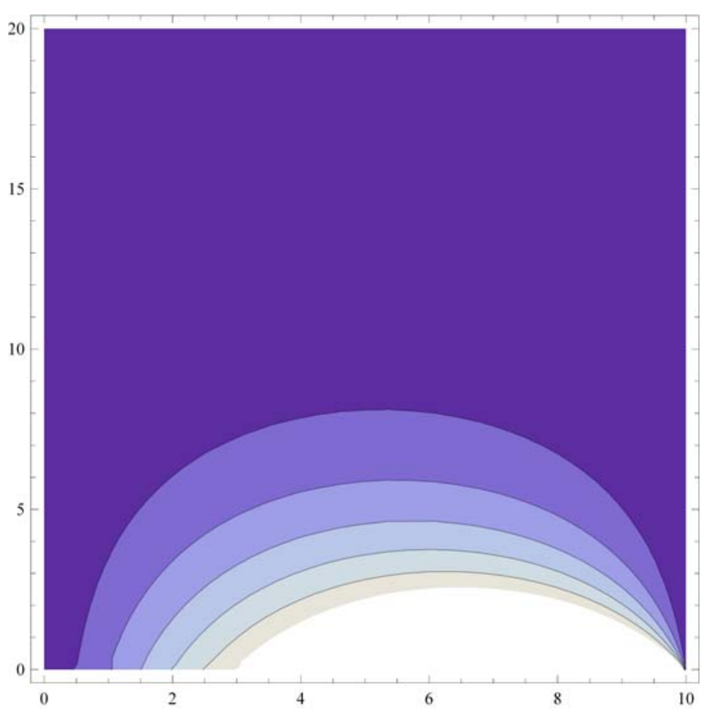

4. Boas, problem 2, page 626.

Solution : Now our width is 20 cm so that the general solution (analogous to eq.(1) in the problem above) becomes :

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n \pi x}{20}\right) e^{-n \pi y/20} \tag{2}$$

and applying the boundary condition at y = 0 gives us :

$$T(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n \pi x}{20}\right) = \begin{cases} 0, & 0 < x < 10 \\ 100, & 10 < x < 20 \end{cases}$$

Again we recognize that we will find the B coefficients by determining the Fourier series that fits the function :

$$f(x) = \begin{cases} -100, & -20 < x < -10 \\ 0, & -10 < x < 0 \\ 0, & 0 < x < 10 \\ 100, & 10 < x < 20 \end{cases}$$

Since this is an odd function, we know only the sin terms will be non - zero, and we solve for the Fourier coefficients :

$$B_n = b_n = \frac{2}{L} \int_0^L f(x) \sin(n \pi x / L)\, dx$$

For this case, L = 20, and f (x) = 0 on (0, 10) and f (x) = 100 on (10, 20), so our relevant integral becomes :

$$B_n = b_n = \frac{2}{20} \int_{10}^{20} 100 \sin(n \pi x / 20)\, dx \;\Rightarrow\; B_n = \begin{cases} \frac{200}{n \pi}, & n \text{ odd} \\ \frac{-400}{n \pi}, & n = 2, 6, 10, \ldots \\ 0, & n = 4, 8, 12, \ldots \end{cases}$$

Substituting this into the initial equation (2) yields the complete solution :

$$T(x, y) = \sum_{n=1}^{\infty} \frac{B_n \sin\left(\frac{n \pi x}{20}\right) e^{-n \pi y/20}}{n}$$

using the expression for B_n from above. Plotting these results, we get:

```
Clear[temp]
temp[x_, y_] := (200 / π) Sum[Sin[n π x / 20] Exp[-n π y / 20] / n, {n, 1, 151, 2}] -
   (400 / π) Sum[Sin[n π x / 20] Exp[-n π y / 20] / n, {n, 2, 151, 4}]
ContourPlot[temp[x, y], {x, 0, 20}, {y, 0, 20}]
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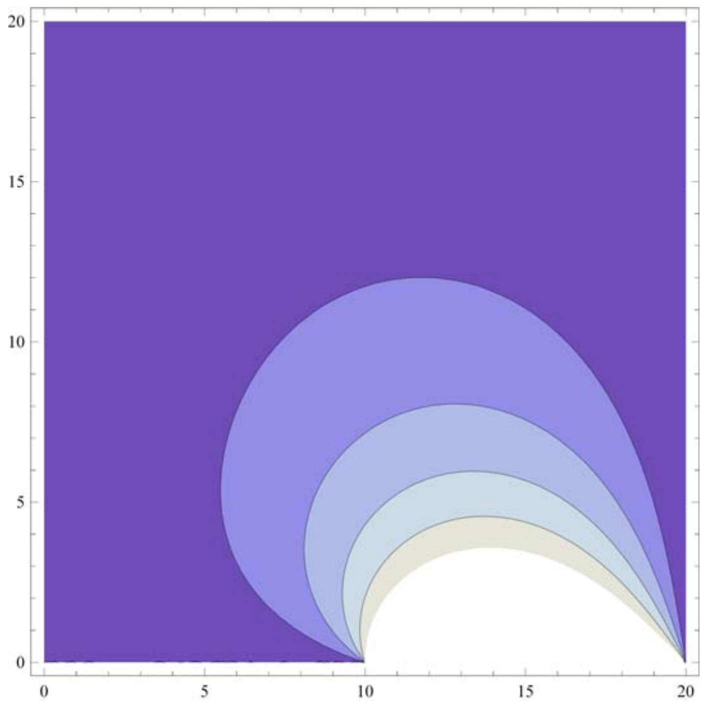

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5. Boas, problem 7, page 626.

Solution : In this case, the width of the plate is π and the height is 1. Since the plate is not infinite in y, we cannot automatically set any coefficients equal to zero; following the treatment on p. 624 of Boas, we know our general solution is :

$$T(x, y) = \sum_{k=1}^{\infty} (A_k \cos k x + B_k \sin kx)\left(Ce^{k y} + De^{-ky}\right)$$

As shown in Boas and as done in class, the boundary condition that T (x, 1) = 0 leads to :

$$Ce^{k y} + De^{-ky} = \frac{1}{2}\left(e^{k(1-y)} - e^{k(1-y)}\right) = \sinh[k(1-y)]$$

Applying the other boundary conditions yields :

• $T(0, y) = 0 \Rightarrow A = 0$
• $T(\pi, y) = 0 \Rightarrow \sin(k\pi) = 0 \Rightarrow k\pi = n\pi \Rightarrow k = n$

so our general solution becomes :

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin(n x) \sinh[n(1-y)]$$

Applying the boundary condition along the lower strip, T (x, 0) = cos x yields :

$$T(x, 0) = \cos x = \sum_{n=1}^{\infty} B_n \sin(n x) \sinh[n]$$

and again we know we must find the Fourier coefficients of the expansion that will satisfy this boundary condition. In this case, we have that the Fourier coefficients are represented by :

$$b_n = B_n \sinh(n) \Rightarrow B_n = \frac{b_n}{\sinh(n)}$$

To find the value of the Fourier coefficients, b_n, we compute the integral:

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(n x)\, dx = \begin{cases} 0, & n \text{ odd} \\ \frac{4}{\pi} \cdot \frac{1}{n^2-1}, & n \text{ even} \end{cases}$$

(Make sure you recognize that we make the odd extension of cos x into the left half plane; we need to make the odd extension since we can see that we have to reproduce the Fourier sine series). This gives us the values for the b_n's; to get the values for B_n:

$$B_n = \frac{b_n}{\sinh(n)} = \frac{4}{\pi} \cdot \frac{1}{\sinh(n)\, n^2 - 1} \quad \text{for n even}$$

our complete solution is then :

$$T(x, y) = \frac{4}{\pi} \sum_{n,\, \text{even}}^{\infty} \frac{\sin(n x) \sinh[n(1-y)]}{\sinh(n)\left(n^2-1\right)}$$

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6. Boas, problem 2, page 632.

This is very similar to example 1 done in Boas (pp. 629 - 630). In the problem we have here, the length of the rod is set to L = 10 cm, and the initial temperature distribution is u (x, 0) = 100. Therefore, we follow the treatment done in class and in Boas, using Boas eq. 3.12 (p. 630) to show that the solution will be of the form :

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-(n \pi \alpha/10)^2 t} \sin\left(\frac{n \pi x}{10}\right) \quad (3)$$

Applying the boundary condition that u (x, 0) = 100, we find :

$$u(x, 0) = 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n \pi x}{10}\right)$$

and we realize that we have the Fourier sine series for f (x) = 100. Solving for the Fourier coefficients :

$$b_n = \frac{2}{10}\int_0^{10} 100\cdot\sin(n\pi x/10)\,dx = \begin{cases} \frac{400}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Substituting these values for b_n into equation (3) yields the complete solution:

$$u(x,t) = \frac{400}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{e^{-(n\pi\alpha/10)^2 t}\sin\left(\frac{n\pi x}{10}\right)}{n}$$

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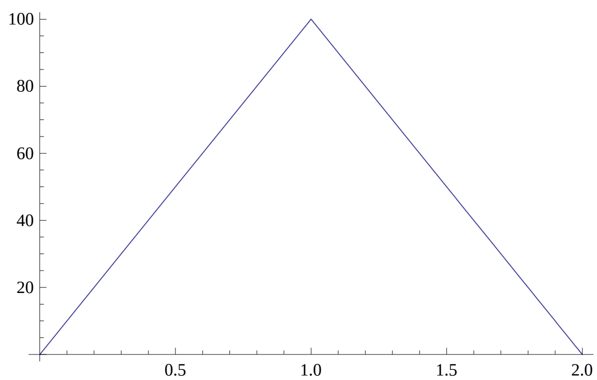
7. Boas, problem 5, page 632.

Solution : This problem follows the procedure described in Example 2 in Boas, pp. 630 - 631. The initial temperature distribution can be described by :

$$u(x,0) = \begin{cases} 100\,x, & 0 < x < 1 \\ 100\,(2-x), & 1 < x < 2 \end{cases}$$

and the function looks like :



The final temperature is 100 across the length of the slab, so (as shown by Boas eq. (3.16)), the general form of the solution is :

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\, e^{-(n\pi\alpha/L)^2 t} + u_f$$

In this specific example, L = 2 and u_f = 100. If we apply the t=0 boundary condition, we can rewrite this solution as:

$$u(x,t) - u_f = \sum_{n=1}^{\infty} b_n \sin(n\pi x/10) \tag{4}$$

and we recognize that we can find the Fourier b coefficients by expanding the left side in a Fourier series. The explicit function on the left we will expand is :

$$u(x,t) - u_f = \begin{cases} 100\,x - 100, & 0 < x < 1 \\ 100\,(2 - x) - 100, & 1 < x < 2 \end{cases}$$

which we can write as :

$$u(x,t) - u_f = \begin{cases} 100\,(x - 1), & 0 < x < 1 \\ 100\,(1 - x), & 1 < x < 2 \end{cases}$$

This is the function we need to expand in a Fourier sine series, and that the calculated b_n coefficients will be substituted into eq. (4). We have then:

$$b_n = \frac{2}{2}\left[\int_0^1 100\,(x-1)\sin(n\pi x/2)\,dx + \int_1^2 100\,(1-x)\sin(n\pi x/2)\,dx\right]$$

$$b_n = \begin{cases} 0, & n \text{ even} \\ \frac{-400\,(-2+n\pi)}{n^2\pi^2}, & n = 1, 5, 9, \ldots \\ \frac{400\,(2+n\pi)}{n^2\pi^2}, & n = 3, 7, 11, \ldots \end{cases}$$

Substituting these values for b into equation (4) will provide the complete solution to the problem.

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8. Boas, problem 1, page 650. (Assume azimuthal symmetry)

Solution : We are asked to find the solution inside a sphere of radius 1 for the surface temperature distribution equal to 35 $\cos^4\theta$. We follow the solution in Boas (pp. 647-649) and in the classnote on solving Laplace’s equation in spherical coordinates. We know the general solution will be of the form:

$$T(r,\ \theta) = \sum_{m=0}^{\infty} \left(A_m r^m + B_m r^{-(m+1)}\right) P_m(\cos\theta)$$

Since we are asked to find the temperature distribution inside the sphere, we know that the B terms must all be zero, otherwise we have an infinite temperature at r = 0. Therefore, our general solution becomes :

$$T(r,\ \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

Now, we apply the boundary condition; since r = 1 we have :

$$T(1,\theta) = \sum_{m=1}^{\infty} A_m P_m(\cos\theta) = 35\cos^4\theta$$

Finding the complete solution requires determining the values of the A_m coefficients, and we can do this by realizing that the sum is merely a Legendre series for the function 35 $\cos^4\theta$. Setting x = cos

θ, we know that the recipe for finding the coefficients of a Legendre series is:

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx = \frac{2m+1}{2} \cdot 2 \int_{0}^{1} x^4 P_m(x)\, dx$$

Since our function, $35 \cos^4 \theta$ is even, we will get non zero coefficients only for the even coefficients; these coefficients are:

$$c_0 = \frac{1}{2} \cdot 2 \int_0^1 x^4 \cdot 1\, dx = \frac{1}{5}$$

$$c_2 = \frac{5}{2} \cdot 2 \int_0^1 \frac{1}{2} \left(3x^2 - 1\right) \cdot x^4\, dx = \frac{4}{7}$$

$$c_4 = \frac{9}{2} \cdot 2 \int_0^1 \frac{1}{8} \cdot \left(3 - 30x^2 + 35x^4\right) \cdot x^4\, dx = \frac{8}{35}$$

and we can write the first three non zero terms of the solution as :

$$T(r, \theta) = \frac{1}{5} + \frac{4}{7} r^2 P_2(\cos\theta) + \frac{8}{35} r^4 P_4(\cos\theta)$$

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9. Boas, problem 8, page 650. (Assume azimuthal symmetry)

Solution : We are asked to find the interior temperature of a sphere of radius 1 whose surface temperature is defined by :

$$f(\theta) = \begin{cases} 100, & 0 < \theta < \pi/3 \\ 0, & \text{otherwise} \end{cases}$$

We know from above that the general solution will be of the form :

$$T(r, \theta) = T(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

Since we are finding the interior temperature, the B coefficients must all be zero otherwise there will be a singularity at the center of the sphere where r = 0. We know apply the bondary condition at the surface; setting $x = \cos\theta$, we can rewrite the boundary condition as :

$$f(x) = \begin{cases} 100, & 1/2 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then setting r = 1 at the surface, we have :

$$T(1, x) = \sum_{m=0}^{\infty} A_m P_m(x) = \begin{cases} 100, & 1/2 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The only unknown remaining to determine a complete solution is the evaluation of the A coefficients. We realize that these are just the coefficients in the Legendre Series expansion of our function; so we can compute them as :

$$A_m \equiv c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, P_m(x)\, dx$$

for this function, the integral becomes :

$$A_m \equiv c_m = \frac{2m+1}{2} \int_{1/2}^{1} 100\, P_m(x)\, dx$$

$$c_0 = \frac{1}{2} \int_{1/2}^{1} 100\, dx = 25$$

$$c_1 = \frac{3}{2} \int_{1/2}^{1} 100\, x\, dx = \frac{225}{4}$$

$$c_2 = \frac{5}{2} \int_{1/2}^{1} 100 \cdot \frac{1}{2} \left(3x^2 - 1\right) dx = \frac{375}{8}$$

$$c_3 = \frac{7}{2} \int_{1/2}^{1} 100 \cdot \frac{1}{2} \left(5x^3 - 3x\right) dx = \frac{525}{64}$$

and the first few terms of the Legendre expansion are :

$$T(r, \theta) = 25\, r^0\, (1) + \frac{225}{4}\, r \cos\theta + \frac{375}{8}\, r^2 \left(\frac{1}{2}\left(3\cos^2\theta - 1\right)\right) + \frac{525}{54}\, r^3 \left(\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right) + \ldots$$