PHYS 301

Homework #1--Solutions

Due : 25 January 2012

For all homework assignments this term, include complete and clear solutions with your answers. Assigned credit will be determined by both the accuracy of your answer and the completeness and clarity of the logic you employ in reaching your final result.

1. Consider a vector **A** of magnitude A making an angle θ above the positive x axis. Consider a vector **B** of magnitude B in the fourth quadrant making an angle ϕ below the positive x axis. Use the definitions and properties of the cross and dot products to show that :

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

We start by noting the standard definitions of the dot and cross product :

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\;|\mathbf{B}|\;\cos(\theta+\phi)$$
$$|\mathbf{B}\times\mathbf{A}| = |\mathbf{B}|\;|\mathbf{A}|\;\sin(\theta+\phi)$$

The dot and cross products depend on the "angle between the vectors"; in this case, that angle is simply $\theta + \phi$. We can also write the components of these vectors a la intro physics :

$$A_x = A\cos\theta; \quad A_y = A\sin\theta; \quad B_x = B\cos\phi; \quad B_y = -B\sin\phi$$

Writing the dot and cross products in terms of components :

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y$$
$$|\mathbf{B}\times\mathbf{A}| = B_x A_y - B_y A_x$$

Now, we equate the component expressions with the standard definitions :

$$|A||B|\;\cos(\theta+\phi) = A\,B\cos\theta\cos\phi - A\,B\sin\theta\;\sin\phi$$
$$|A|\;|B|\;\sin(\theta+\phi) = A\,B\cos\phi\sin\theta - A\,B\cos\theta\,(-\sin\phi)$$

Dividing through by the common term A B and rearranging, we get :

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$
$$\sin(\theta + \phi) = \sin\theta \cos\phi + \sin\phi \cos\theta$$

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2. Consider the scalar function :

$$\phi = x\,y\,z + x^2 y^2 z^2$$

a) Find the gradient of ϕ ; i.e., $\nabla\phi$.

The gradient is :

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}} =$$
$$\left(y\,z + 2\,x\,y^2 z^2\right)\hat{\mathbf{x}} + \left(x\,z + 2\,x^2 y\,z^2\right)\hat{\mathbf{y}} + \left(x\,y + 2\,x^2 y^2 z\right)\hat{\mathbf{z}}$$

b) Find the divergence of $\nabla\phi$, i.e., $\nabla\cdot\nabla\phi$

The divergence of the gradient is :

$$\frac{\partial}{\partial x}\left(y\,z + 2\,x\,y^2 z^2\right) + \frac{\partial}{\partial y}\left(x\,z + 2\,x^2 y\,z^2\right) + \frac{\partial}{\partial z}\left(x\,y + 2\,x^2 y^2 z\right) =$$

$$2\,y^2 z^2 + 2\,x^2 y^2 + 2\,x^2 z^2$$

c) Find the curl of $\nabla\phi$, i.e., $\nabla\times(\nabla\phi)$

Computing the components of the curl, you should find that the curl is zero.We will show later that $\nabla\times\nabla\phi = 0$ for all scalar potentials ϕ.

Verifying these results via Mathematica

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<< VectorAnalysis`
Clear[phi, x, y, z]
phi = x y z + (x y z)^2;
gradphi = Grad[phi, Cartesian[x, y, z]]
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$\{y\,z + 2\,x\,y^2 z^2,\ x\,z + 2\,x^2 y\,z^2,\ x\,y + 2\,x^2 y^2 z\}$

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Div[gradphi, Cartesian[x, y, z]]
```

$2\,x^2 y^2 + 2\,x^2 z^2 + 2\,y^2 z^2$

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Curl[gradphi, Cartesian[x, y, z]]
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{0, 0, 0}
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3. Boas, p. 307, no. 4 parts a) and b) . Ten points for each part.

a) We are asked to compute the line integral for :

$$\int_C y^2\,dx + 2\,x\,dy + dz$$

where C is the contour consisting of a line from (0, 0, 0) to (1, 0, 0); a line from (1, 0, 0) to (1, 0, 1) and finally from (1, 0, 1) to (1, 1, 1).

For this contour, we break the line integral into three parts; for the first segment, running from (0, 0, 0) to (1, 0, 0), we set x = t so that dx = dtand also y = 0 = z, therefore, the integral becomes :

$$\int_1 y^2\,dx + 2\,x\,dy + dz = 0 \sin y = dy = dz = 0$$

Along the segment from (1, 0, 0) to (1, 0, 1), x = 1, y = 0, z = t (so dz = dt) and we have :

$$\int_2 y^2\,dx + 2\,x\,dy + dz = \int_0^1 dt = 1$$

Along the final segment from (1, 0, 1) to (1, 1, 1), x = 1 = z, y = t (dy = dt) and :

$$\int_3 y^2\,dx + 2\,x\,dy + dz = \int_0^1 t\,(0) + 2\,(1)\,dt + 0 = 2$$

(Remember that if x = 1, then dx = 0.) The final integral is the sum of these three segments, so we have in total :

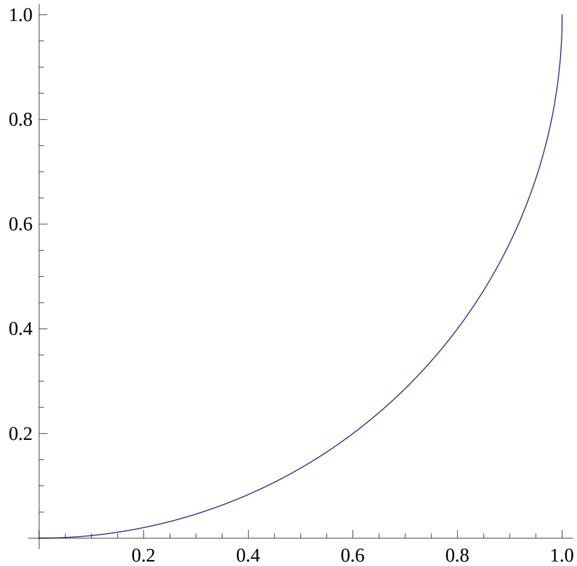
$$\int_C y^2\,dx + 2\,x\,dy + dz = 3$$

b) here, our contour consists of the circular arc from the origin to (1, 1, 0) along the arc defined by $x^2 + y^2 - 2\,y = 0$ then the straight line from (1,1,0) to (1,1,1).

Computing first the contribution to the line integral along the arc, we notice that we can write the defining relationship as

$$x^2 + (y-1)^2 - 1 = 0 \Rightarrow x^2 + (y-1)^2 = 1$$

This represents an arc along the circle of radius 1 centered at (0, 1) :



Looking at the curve, notice that our limits of integration will be from - $\pi/2$ to 0. Since our curve is a circle, we can parameterize as :

$$x = \cos\theta; \quad y - 1 = \sin\theta; \ z = 0$$
$$dx = -\sin\theta\, d\theta; \ \ dy = \cos\theta\, d\theta; dz = 0$$

The integral becomes :

$$\int_{-\pi/2}^{0} \left\{(\sin\theta + 1)^2 (-\sin\theta)\, d\theta + 2\cos\theta\,(\cos\theta\, d\theta)\right\}$$

Evaluating this integral :

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Integrate[-Sin[θ] (Sin[θ] + 1) ^2 + 2 Cos[θ] ^2, {θ, -π / 2, 0}]
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$\frac{5}{3}$

This is the contribution to the line integral from the arc. Along the straight line from (1, 1, 0) to (1, 1, 1) we have that dx = dy = 0, so that the line integral reduces to :

$$\int_C y^2\, dx + 2\,x\, dy + dz = \int_0^1 dz = 1$$

Therefore, the total line integral is 1 + 5/3 = 8/3.

4. Evaluate numerically

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im}$$

where the δ are Kronecker deltas.

We make successive use of the contraction property of Kronecker deltas :

$$\delta_{\alpha\beta}\,\delta_{\beta\gamma} = \delta_{\alpha\gamma}$$

Thus,

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im} = \delta_{ik}\,\delta_{km}\,\delta_{im} = \delta_{im}\,\delta_{im} = \delta_{im}\,\delta_{mi} = \delta_{ii} = \delta_{mm} = 3$$