# PHYS301

# HOMEWORK #3

Due : Monday, 13 Feb. 2012

1. Verify the divergence theorem for the function :

$$\mathbf{v} \;=\; (x\,y)\,\hat{\mathbf{x}} \;+\; (2\,y\,z)\,\hat{\mathbf{y}} \;+\; (3\,z\,x)\,\hat{\mathbf{z}}$$

for the volume defined by the cube of length 2 in the first octant (meaning one vertex is at (0, 0, 0) and the other is at (2, 2, 2)).

***Solution :***

We verify the divergence theorem by computing both :

$$\int_V \nabla \cdot \mathbf{v}\, d\tau \quad \text{and} \int_S \mathbf{v} \cdot\, d\mathbf{a}$$

Doing the volume integral first :

$$\nabla \cdot \mathbf{v} \;=\; y \;+\; 2\,z \;+\; 3\,x$$

so that the volume integral becomes :

$$\int_V \nabla \cdot \mathbf{v}\, d\tau \;=\; \int_0^2 \int_0^2 \int_0^2 (3\,x + y + 2\,z)\, dx\, dy\, dz \;=\; \int_0^2 \int_0^2 (6 + 2\,y \;+ 4\,z)\, dy\, dz$$

$$= \;\; \int_0^2 (12 + \;4 + 8\,z)\, dz \;=\; 24 + 8 + 16 \;=\; 48$$

The surface integral requires 6 separate integrations :

1. For the face at x = 2 whose outward normal is + $\hat{\boldsymbol{x}}$:

$$\int_S \mathbf{v} \cdot d\mathbf{a} \;=\; \int_S x\,y\, dy\, dz \;=\; 2 \int_0^2 \int_0^2 y\, dy\, dz \;=\; 8$$

2. For the face at x = 0 the surface integral is zero since x = 0.

3. For the face at y = 2 whose outward normal is + $\hat{\boldsymbol{y}}$:

$$\int_S \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 2\, y\, z\, dx\, dz = 4 \int_0^2 \int_0^2 z\, dx\, dz = 16$$

4. For the face at y = 0, the surface integral is zero since y = 0.

5. For the face at z = 2 whose outward normal is + $\hat{\mathbf{z}}$ **:**

$$\int_S \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 3\, z\, x\, dx\, dy = 6 \int_0^2 \int_0^2 x\, dx\, dy = 24$$

6. For the face at z = 0, the surface integral is zero since z = 0.

The sum of these surface integrals is 8 + 16 + 24 = 48, and we verify the divergence theorem

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2. Verify Stokes' Theorem for the function :

$$\mathbf{v} = a\, y\, \hat{\mathbf{x}} + b\, x\, \hat{\mathbf{y}}$$

over the circle of radius R centered at the origin, and a and b are constants.

Solution :

We first solve the surface integral of the curl :

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_S (b - a)\, \hat{\mathbf{z}} \cdot da\, \hat{\mathbf{z}} = (b - a) \int_S da$$

Where we determine that curl $\mathbf{v}$ = (b - a)in the $\hat{\mathbf{z}}$ direction which lies parallel to the unit normal outward from the x-y plane. Therefore, (b - a) is a constant and can be taken out of the integral leaving us to find the surface area of the circle of radius R, this immediately yields the solution :

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = (b - a)\, \pi\, R^2$$

To find the line integral, we parameterize the circle as :

$$x = R\cos\theta \quad dx = -R\sin\theta\, d\theta$$
$$y = R\sin\theta \quad dy = R\cos\theta\, d\theta$$

therefore, the line integral becomes :

$$\int_C \mathbf{v}\cdot d\mathbf{l} = \int_C v_x\, dx + v_y\, dy = \int (a\,R\sin\theta)\,(-R\sin\theta\, d\theta) + (b\,R\cos\theta)\,(R\cos\theta\, d\theta) =$$

$$\int_0^{2\pi} R^2\left(b\cos^2\theta - a\sin^2\theta\right) d\theta = \pi R^2\,(b-a)$$

$$\text{since } \int_0^{2\pi} \sin^2 x\, dx = \int_0^{2\pi} \cos^2 x\, dx = \pi$$

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3. Compute the line integral of :

$$\mathbf{v} = 6\,\hat{\mathbf{x}} + y z^2\,\hat{\mathbf{y}} + (3y + z)\,\hat{\mathbf{z}}$$

over the triangular path defined by the line segments (0, 0, 2) -> (0, 0, 0) -> (0, 1, 0) -> (0, 0, 2).

Solution :

We can solve this either via direct computation of the line integral, or by using Stokes' Theorem. If we use Stokes' Theorem, we notice that the unit normal to the plane of the triangle is in the + $\hat{\boldsymbol{x}}$ direction, so the only contribution to the surface integral:

$$\int_S (\nabla\times\mathbf{v})\cdot d\mathbf{a}$$

will come from the $\hat{x}$ direction of the curl; taking the curl we obtain:

$$\int_S (\nabla\times\mathbf{v})\cdot da = \int_0^1 \int_0^{-2y+2} (3 - 2yz)\, dz\, dy$$

The area we are integrating over is the triangle defined by the y axis, z axis, and the line z = - 2 y + 2. This integral becomes :

$$\int_0^1 \int_0^{-2\,y+2} (3 - 2\,yz)\,dz\,dy \;=\; \int_0^1 \left(3\,z \;-\; yz^2\right)\Big|_0^{-2\,y+2} dy \;=$$

$$\int_0^1 \left\{3\,(-2\,y + 2) \;-\, y\,(-2\,y + 2)^2\right\} dy \;=\; \int_0^1 \left(-4\,y^3 + 8\,y^2 - 10\,y + 6\right) dy \;=\; \frac{8}{3}$$

The problem does not ask us to verify Stokes' Theorem, so we could stop here, but for completeness, let' s compute the line integral :

$$\int_C \mathbf{v} \cdot d\mathbf{l} \;=\; \int_C v_x\; dx \;+\; v_y\,dy \;+ v_z\,dz$$

The first path is along the z axis from z = 2 to z = 0, so the only contribution to the integral comes from the term :

$$\int_1 \mathbf{v} \cdot d\mathbf{l} = \int_2^0 v_z\,dz \;=\; \int_2^0 (3\,y + z)\,dz \;\;=\; \int_2^0 z\,dz \;=\; -\,2$$

Where the last integral derives from the fact that y = 0 along this line. For the second path, from (0, 0, 0) to (0, 1, 0); the only contribution to this path could come from the $v_y dy$ term, and we have:

$$\int_2 \mathbf{v} \cdot dl \;=\; \int_0^1 y\,z^2\,dy \;=\; 0 \text{ since } z \;= 0 \text{ along this path.}$$

Finally, we must parameterize along the line from (0, 1, 0) to (0, 0, 2). The equation of this line is z = **-** 2 y + 2, so our parameterization becomes :

$$y \;=\; t;\;\; dy \;=\; dt$$

$$z \;=\; -\,2\,t + 2;\;\; dz \;=\; -\,2\,dt$$

$$\int_3 \mathbf{v} \,\cdot\, d\mathbf{l} \;=\; \int_C v_x\;\, dx \;+\; v_y\,dy \;=\; \int_1^0 \left\{t\,(-2\,t + 2)^2\,dt \;+ (3\,t + (-2\,t + 2))\,(-2\,dt)\right\}$$

Simplifyng the integrand yields :

$$\int_3 \mathbf{v} \cdot dl = \int_1^0 \left(4\,t^3 - 8\,t^2 + 2\,t - 4\right) dt = \frac{14}{3}$$

Note the limits of integration; since the path starts from (0, 1, 0) going to (0, 0, 2), our parameterization of $z = -2\,t + 2$ shows that when $t = 1$, $z = 0$ and that $t = 0 \Rightarrow z = 2$.

Then, the total line integral is the sum of all three paths, and equals :

$$-2 + 0 + \frac{14}{3} = \frac{8}{3}$$ as we found using Stokes' Theorem.

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4. Write three short Mathematica programs showing how to compute the value of 10! using a Do loop, For statement and While statment

***Solutions :***

For the Do loop, we set the initial value of the variable fact = 1 and :

```
In[1]:= Clear[fact, n]
        fact = 1;
        Do[fact = n fact, {n, 10}]
        Print[fact]
```

```
3 628 800
```

For the while statement :

```
In[8]:= Clear[fact, n]
        fact = 1; n = 1; While[n < 11, fact = n fact; n++]
        Print[fact]
```

```
3 628 800
```

For the For statement :

```
In[15]:= Clear[fact, n]
         fact = 1;
         For[n = 1, n < 11, n++, fact = n fact]
         Print[fact]
```

```
3 628 800
```