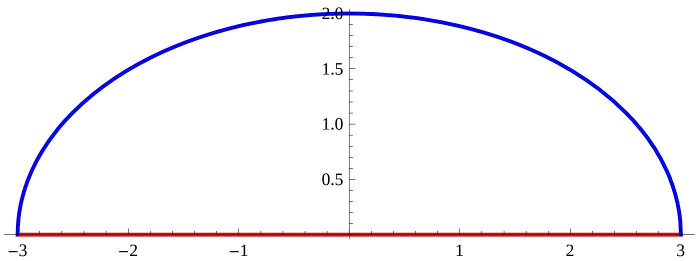
PHYS 301
HW 4 -- SOLUTIONS

1. We are asked to find the line integral of a rather intricate function over the upper branch of an ellipse. Brute force integration is possible but will be lengthy; the hint suggests using Stokes' Theorem. Many students asked me how this is possible since Stokes' Theorem requires a closed path; the answer is to create one and see if this approach makes your life easier. In particular, I choose the path outlined below; the path that includes the upper portion of the ellipse (in blue) and the straight line (in red) from x = -3 to x = 3.



Now I have a contour over which I can calculate the line integral. In using Stokes' Theorem, I must compute the curl of the vector field to compute the flux of the curl through the surface. Since this surface is in the x - y plane, the unit normal to this surface is in the + **z** direction, so I need to find the **z** component of the curl of **F**. Doing so, I find :

```
In[13]:= Needs["VectorAnalysis`"]
Clear[f]
f = {Exp[x] Cos[y] - Exp[-x] Sin[y], -Exp[x] Sin[y] + Exp[-x] Cos[y], 0};
Curl[f, Cartesian[x, y, z]]

Out[16]= {0, 0, 0}
```

This function was carefully chosen to produce a curl of zero. Since the curl of the function is zero, we know that the line integral over the entire path is zero. We can write the line integral as a sum of its components as :

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_E \mathbf{F} \cdot d\mathbf{l} + \int_L \mathbf{F} \cdot d\mathbf{l} = 0$$

where C is the total contour, E is the portion of the contour along the upper portion of the ellipse, and L is the path along the x axis from x = -3 to x = 3. Since the line integral of a conservative function is zero, we know that

$$\int_E \mathbf{F} \cdot d\mathbf{l} = -\int_L \mathbf{F} \cdot d\mathbf{l}$$

Thus, our problem reduces to solving the line integral along the x axis. We can parameterize this line integral as :

$x = t \Rightarrow dx = dt$

$y = 0 = dy$

So the line integral along the x axis becomes :

$$\int_L \mathbf{F} \cdot d\mathbf{l} = \int_L F_x\, dx + F_y\, dy = \int_L F_x\, dx = \int_{-3}^{3} (e^x \cos y - e^{-x} \sin y)\, dx$$

since y = 0 = dy along this path. Along the x axis, cos y = 1 and sin y = 0, so that the line integral further simplifies to :

$$\int_{-3}^{3} e^x\, dx = e^3 - e^{-3} = 2 \sinh 3$$

It is then trivial to argue that the line integral along the upper portion of the ellipse is simply - 2 sinh 3.

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2. Starting with the divergence theorem :

$$\int_V \nabla \cdot \mathbf{v}\, d\tau = \int_S \mathbf{v} \cdot d\mathbf{a}$$

If $\mathbf{v} \to \mathbf{v} \times \mathbf{c}$, then :

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{c})\, d\tau = \int_S (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a}$$

We can use summation notation to show that the integrand on the left becomes :

$$\nabla \cdot (\mathbf{v} \times \mathbf{c}) \to \frac{\partial}{\partial x_i} \left(\epsilon_{ijk}\, v_j\, c_k\right) = \epsilon_{ijk} \frac{\partial}{\partial x_i} v_j\, c_k = v_j\, \epsilon_{ijk} \frac{\partial}{\partial x_i} c_k + c_k\, \epsilon_{ijk} \frac{\partial}{\partial x_i} v_j$$

From the last equality above, we obtain $- \mathbf{v} \cdot (\nabla \times \mathbf{c}) + \mathbf{c} \cdot (\nabla \times \mathbf{v})$. Since **c** is a constant vector, curl **c** = 0, so the integrand on the left side of the Divergence Theorem is just $\mathbf{c} \cdot (\nabla \times \mathbf{v})$.

On the right side of the Divergence Theorem, the integrand represents the volume of a parallelogram defined by the vectors **v**, **c** and d**a**. Therefore (as shown in class earlier this semester and is described in the classnote on the epsilon delta relationship),

$$(\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a} = (\mathbf{c} \times d\mathbf{a}) \cdot \mathbf{v} = (d\mathbf{a} \times \mathbf{v}) \cdot \mathbf{c} = - \mathbf{c} \cdot (\mathbf{v} \times d\mathbf{a})$$

Therefore, we can rewrite the Divergence theorem as :

$$\int_V \mathbf{c}\cdot(\nabla\times\mathbf{v})\,d\tau = -\int_S \mathbf{c}\cdot(\mathrm{v}\times d\mathbf{a})$$

Take the dot product of both sides with c, then divide by c^2 which will produce the final result needed:

$$\int_V (\nabla\times\mathbf{v})\,d\tau = -\int_S \mathbf{v}\times d\mathbf{a}$$

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3. To find the scale factors, we use the transformation equatons and write expressions for dx, dy, and dz :

$$dx = a\sinh u\cos v\,du - a\cosh u\sin v\,dv$$
$$dy = a\cosh u\sin v\,du + a\sinh u\cos v\,dv$$
$$dz = dz$$

Since we know the length of an infinitesimally small increment is the same in all coordinate systems we find expressions for :

$$(dx)^2 = a^2\sinh^2 u\cos^2 v\,(du)^2 + a^2\cosh^2 u\sin^2 v\,(dv)^2 - 2a^2\sinh u\cosh u\cos v\sin v\,du\,dv$$
$$(dy)^2 = a^2\cosh^2 u\sin^2 v\,(du)^2 + a^2\sinh^2 u\cos^2 v\,(dv)^2 + 2a^2\cosh u\sinh u\cos v\sin v\,du\,dv$$
$$(dz)^2 = (dz)^2$$

Adding terms, we see the cross terms (the du dv terms) sum to zero, leaving us with :

$$(dx)^2 + (dy)^2 = a^2\left[(du)^2\left\{\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v\right\} + (dv)^2\left\{\cosh^2 u\sin^2 v + \sinh^2 u\cos^2 v\right\}\right]$$

Where I have omitted the square of the dz term since that is trivial to add on at the end. We can make use of the standard trig and hyperbolic relationships to simplify the equations above. Using

$$\cosh^2 t - \sinh^2 t = 1 \Rightarrow \cosh^2 t = 1 + \sinh^2 t$$

and :

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \cos^2 t = 1 - \sin^2 t$$

we find that the du term becomes :

$$a^2\left[\sinh^2 u\left(1-\sin^2 v\right) + \left(1+\sinh^2 u\right)\sin^2 v\right] = a^2\left[\sinh^2 u + \sin^2 v\right]$$

the dv term becomes :

$$a^2\left[\left(1+\sinh^2 u\right)\sin^2 v + \sinh^2 u\left(1-\sin^2 v\right)\right] = a^2\left[\sin^2 v + \sinh^2 u\right]$$

We find the scale factors by taking the square root of the $(du)^2$and $(dv)^2$:

$$h_1 = h_u = a\sqrt{\sinh^2 u + \sin^2 v}$$
$$h_2 = h_v = a\sqrt{\sinh^2 u + \sin^2 v}$$

Finally, it should be clear that :

$$h_3 = h_z = 1$$

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4. We begin by find expressions for dx, dy and dz :

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos$

$dx = \sin\theta \cos\phi \, dr + r \cos\theta \cos\phi \, d\theta - r \sin\theta \sin\phi \, d\phi$

$dy = \sin\theta \sin\phi \, dr + r \cos\theta \sin\phi \, d\theta + r \sin\theta \cos\phi \, d\phi$

$z = \cos\theta \, dr - r \sin\theta \, d\theta$

$(dx)^2 = \sin^2\theta \cos^2\phi \, (dr)^2 + r^2 \cos^2\theta \cos^2\phi \, (d\theta)^2 + r^2 \sin^2\theta \sin^2\phi \, (d\phi)^2 +$
$\quad 2 r \sin\theta \cos\theta \cos^2\phi \, dr \, d\theta - 2 r \sin^2\theta \cos\phi \sin\phi \, dr \, d\phi - 2 r^2 \sin\theta \cos\theta \sin\phi \cos\phi \, d\theta \, d\phi$

$(dy)^2 = \sin^2\theta \sin^2\phi \, (dr)^2 + r^2 \cos^2\theta \sin^2\phi \, (d\theta)^2 + r^2 \sin^2\theta \cos^2\phi \, (d\phi)^2 +$
$\quad 2 r \sin\theta \cos\theta \sin^2\phi \, dr \, d\theta + 2 r \sin^2\theta \sin\phi \cos\phi \, dr \, d\phi + 2 r^2 \sin\theta \cos\theta \sin\phi \cos\phi \, d\theta \, d\phi$

$(dz)^2 = \cos^2\theta \, (dr)^2 + r^2 \sin^2\theta \, (d\theta)^2 - 2 r \sin\theta \cos\theta \, dr \, d\theta$

$(dx)^2 + (dy)^2 + (dz)^2 = (dr)^2\left[\sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta\right] +$
$\quad (d\theta)^2\left[r^2 \left\{\cos^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + \sin^2\theta\right\}\right] + (d\phi)^2\left[r^2 \left\{\sin^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi\right\}\right]$
$+ dr \, d\theta\left[2 r \left\{\sin\theta \cos\theta \cos^2\phi + \sin\theta \cos\theta \sin^2\phi - \sin\theta \cos\theta\right\}\right] +$
$\quad dr \, d\phi \left[2 r \left\{- \sin^2\theta \cos\phi \sin\phi + \sin^2\theta \sin\phi \cos\phi\right\}\right] +$
$\quad d\theta \, d\phi\left[2 r^2 \{-\sin\theta \cos\theta \sin\phi \cos\phi + \sin\theta \cos\theta \sin\phi \cos\phi\}\right]$

We can see that the cross terms ($dr\, d\theta$, $dr\, d\phi$ and $d\theta\, d\phi$) sum to zero, and after applying basic trig identities and grouping terms we can determine that :

the $(dr)^2$ term becomes $(dr)^2\left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right] = (dr)^2$

the $(d\theta)^2$ term : $(d\theta)^2\left[r^2 \left\{\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \sin^2\theta\right\}\right] = r^2 \, (d\theta)^2$

the $(d\phi)^2$ term becomes : $(d\phi)^2\left[r^2 \sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)\right] = r^2 \sin^2\theta \, (d\phi)^2$

The scale factors are just the square roots of the coefficients of these terms, so we have finally :

$h_1 = h_r = 1; \quad h_2 = h_\theta = r; \quad h_3 = h_\phi = r \sin\theta$

For instance, these scale factors imply that the element of length in spherical polar coordinates is :

$dl = h_i \, dq_i \, \hat{q}_1 = dr \, \hat{r} + r \, d\theta \, \hat{\theta} + r \sin\theta \, d\phi \, \hat{\phi}$

and an element of volume is $d\tau = r\,(r \sin\theta)\, dr\, d\theta\, d\phi = r^2 \sin\theta \, dr\, d\theta\, d\phi$.

To find the unit vectors, we start by writing the position vector :

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$$

And we find the unit vectors :

$$\hat{\mathbf{r}} = \frac{\partial\mathbf{r}/\partial r}{|\partial\mathbf{r}/\partial r|} = \frac{\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}} =$$

$$\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = \frac{r\cos\theta\cos\phi\,\hat{\mathbf{x}} + r\cos\theta\sin\phi\,\hat{\mathbf{y}} - r\sin\theta\,\hat{\mathbf{z}}}{\sqrt{r^2\left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)}} =$$

$$\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = \frac{\partial\mathbf{r}/\partial\phi}{|\partial\mathbf{r}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\,\hat{\mathbf{x}} + r\sin\theta\cos\phi\,\hat{\mathbf{y}}}{\sqrt{r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$$

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5. To print out the prime Fibonacci numbers among the first 200 Fibonacci numbers::

In[3]:=
```
Do[If[PrimeQ[Fibonacci[n]], Print[n, "    ", Fibonacci[n]]], {n, 200}]
```

```
3      2
4      3
5      5
7      13
11      89
13      233
17      1597
23      28 657
29      514 229
43      433 494 437
47      2 971 215 073
83      99 194 853 094 755 497
131      1 066 340 417 491 710 595 814 572 169
137      19 134 702 400 093 278 081 449 423 917
```

In the homework, I actually asked for the non prime Fibonacci numbers among the first 200 Fibonacci numbers; this can be done by changing the If statement slightly :

In[4]:=

```
Do[If[PrimeQ[Fibonacci[n]], , Print[n, "    ", Fibonacci[n]]], {n, 200}]
```

The difference might be subtle; notice that after "If[PrimeQ[Fibonacci[n]]" there are two commas. Remember that the structure of an If statement is "test, true, false". In this case, we want to print numbers when the statement is false, that is, when the nth Fibonacci number is not prime. Notice that there is only a blank after the first comma; this instructs the program to do nothing if the statement is true, and only print if the statement is false. We obtain the following data :

```
1    1
2    1
6    8
8    21
9    34
10    55
12    144
14    377
15    610
16    987
18    2584
19    4181
20    6765
21    10 946
22    17 711
24    46 368
25    75 025
26    121 393
27    196 418
28    317 811
30    832 040
31    1 346 269
32    2 178 309
33    3 524 578
34    5 702 887
35    9 227 465
36    14 930 352
37    24 157 817
38    39 088 169
```

39	63 245 986
40	102 334 155
41	165 580 141
42	267 914 296
44	701 408 733
45	1 134 903 170
46	1 836 311 903
48	4 807 526 976
49	7 778 742 049
50	12 586 269 025
51	20 365 011 074
52	32 951 280 099
53	53 316 291 173
54	86 267 571 272
55	139 583 862 445
56	225 851 433 717
57	365 435 296 162
58	591 286 729 879
59	956 722 026 041
60	1 548 008 755 920
61	2 504 730 781 961
62	4 052 739 537 881
63	6 557 470 319 842
64	10 610 209 857 723
65	17 167 680 177 565
66	27 777 890 035 288
67	44 945 570 212 853
68	72 723 460 248 141
69	117 669 030 460 994
70	190 392 490 709 135
71	308 061 521 170 129
72	498 454 011 879 264
73	806 515 533 049 393
74	1 304 969 544 928 657
75	2 111 485 077 978 050
76	3 416 454 622 906 707

77	5 527 939 700 884 757
78	8 944 394 323 791 464
79	14 472 334 024 676 221
80	23 416 728 348 467 685
81	37 889 062 373 143 906
82	61 305 790 721 611 591
84	160 500 643 816 367 088
85	259 695 496 911 122 585
86	420 196 140 727 489 673
87	679 891 637 638 612 258
88	1 100 087 778 366 101 931
89	1 779 979 416 004 714 189
90	2 880 067 194 370 816 120
91	4 660 046 610 375 530 309
92	7 540 113 804 746 346 429
93	12 200 160 415 121 876 738
94	19 740 274 219 868 223 167
95	31 940 434 634 990 099 905
96	51 680 708 854 858 323 072
97	83 621 143 489 848 422 977
98	135 301 852 344 706 746 049
99	218 922 995 834 555 169 026
100	354 224 848 179 261 915 075
101	573 147 844 013 817 084 101
102	927 372 692 193 078 999 176
103	1 500 520 536 206 896 083 277
104	2 427 893 228 399 975 082 453
105	3 928 413 764 606 871 165 730
106	6 356 306 993 006 846 248 183
107	10 284 720 757 613 717 413 913
108	16 641 027 750 620 563 662 096
109	26 925 748 508 234 281 076 009
110	43 566 776 258 854 844 738 105
111	70 492 524 767 089 125 814 114
112	114 059 301 025 943 970 552 219
113	184 551 825 793 033 096 366 333

114	298 611 126 818 977 066 918 552
115	483 162 952 612 010 163 284 885
116	781 774 079 430 987 230 203 437
117	1 264 937 032 042 997 393 488 322
118	2 046 711 111 473 984 623 691 759
119	3 311 648 143 516 982 017 180 081
120	5 358 359 254 990 966 640 871 840
121	8 670 007 398 507 948 658 051 921
122	14 028 366 653 498 915 298 923 761
123	22 698 374 052 006 863 956 975 682
124	36 726 740 705 505 779 255 899 443
125	59 425 114 757 512 643 212 875 125
126	96 151 855 463 018 422 468 774 568
127	155 576 970 220 531 065 681 649 693
128	251 728 825 683 549 488 150 424 261
129	407 305 795 904 080 553 832 073 954
130	659 034 621 587 630 041 982 498 215
132	1 725 375 039 079 340 637 797 070 384
133	2 791 715 456 571 051 233 611 642 553
134	4 517 090 495 650 391 871 408 712 937
135	7 308 805 952 221 443 105 020 355 490
136	11 825 896 447 871 834 976 429 068 427
138	30 960 598 847 965 113 057 878 492 344
139	50 095 301 248 058 391 139 327 916 261
140	81 055 900 096 023 504 197 206 408 605
141	131 151 201 344 081 895 336 534 324 866
142	212 207 101 440 105 399 533 740 733 471
143	343 358 302 784 187 294 870 275 058 337
144	555 565 404 224 292 694 404 015 791 808
145	898 923 707 008 479 989 274 290 850 145
146	1 454 489 111 232 772 683 678 306 641 953
147	2 353 412 818 241 252 672 952 597 492 098
148	3 807 901 929 474 025 356 630 904 134 051
149	6 161 314 747 715 278 029 583 501 626 149
150	9 969 216 677 189 303 386 214 405 760 200
151	16 130 531 424 904 581 415 797 907 386 349

152	26 099 748 102 093 884 802 012 313 146 549
153	42 230 279 526 998 466 217 810 220 532 898
154	68 330 027 629 092 351 019 822 533 679 447
155	110 560 307 156 090 817 237 632 754 212 345
156	178 890 334 785 183 168 257 455 287 891 792
157	289 450 641 941 273 985 495 088 042 104 137
158	468 340 976 726 457 153 752 543 329 995 929
159	757 791 618 667 731 139 247 631 372 100 066
160	1 226 132 595 394 188 293 000 174 702 095 995
161	1 983 924 214 061 919 432 247 806 074 196 061
162	3 210 056 809 456 107 725 247 980 776 292 056
163	5 193 981 023 518 027 157 495 786 850 488 117
164	8 404 037 832 974 134 882 743 767 626 780 173
165	13 598 018 856 492 162 040 239 554 477 268 290
166	22 002 056 689 466 296 922 983 322 104 048 463
167	35 600 075 545 958 458 963 222 876 581 316 753
168	57 602 132 235 424 755 886 206 198 685 365 216
169	93 202 207 781 383 214 849 429 075 266 681 969
170	150 804 340 016 807 970 735 635 273 952 047 185
171	244 006 547 798 191 185 585 064 349 218 729 154
172	394 810 887 814 999 156 320 699 623 170 776 339
173	638 817 435 613 190 341 905 763 972 389 505 493
174	1 033 628 323 428 189 498 226 463 595 560 281 832
175	1 672 445 759 041 379 840 132 227 567 949 787 325
176	2 706 074 082 469 569 338 358 691 163 510 069 157
177	4 378 519 841 510 949 178 490 918 731 459 856 482
178	7 084 593 923 980 518 516 849 609 894 969 925 639
179	11 463 113 765 491 467 695 340 528 626 429 782 121
180	18 547 707 689 471 986 212 190 138 521 399 707 760
181	30 010 821 454 963 453 907 530 667 147 829 489 881
182	48 558 529 144 435 440 119 720 805 669 229 197 641
183	78 569 350 599 398 894 027 251 472 817 058 687 522
184	127 127 879 743 834 334 146 972 278 486 287 885 163
185	205 697 230 343 233 228 174 223 751 303 346 572 685
186	332 825 110 087 067 562 321 196 029 789 634 457 848
187	538 522 340 430 300 790 495 419 781 092 981 030 533

188	871 347 450 517 368 352 816 615 810 882 615 488 381
189	1 409 869 790 947 669 143 312 035 591 975 596 518 914
190	2 281 217 241 465 037 496 128 651 402 858 212 007 295
191	3 691 087 032 412 706 639 440 686 994 833 808 526 209
192	5 972 304 273 877 744 135 569 338 397 692 020 533 504
193	9 663 391 306 290 450 775 010 025 392 525 829 059 713
194	15 635 695 580 168 194 910 579 363 790 217 849 593 217
195	25 299 086 886 458 645 685 589 389 182 743 678 652 930
196	40 934 782 466 626 840 596 168 752 972 961 528 246 147
197	66 233 869 353 085 486 281 758 142 155 705 206 899 077
198	107 168 651 819 712 326 877 926 895 128 666 735 145 224
199	173 402 521 172 797 813 159 685 037 284 371 942 044 301
200	280 571 172 992 510 140 037 611 932 413 038 677 189 525