PHYS 301
HOMEWORK #5

Due : 2 March 2012

All work for this homework assignment must be done by hand; Mathematica may be used to verify calculations but should not be the primary source of computations.

1. Using as starting points the expressions you already have for the coordinate transformations, scale factors and unit vectors for the spherical polar coordinate system :

a) Express the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ in terms of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$, and then use this information to write the position vector completely in terms of spherical polar coordinates. You should find that:

$$\mathbf{r} = r\,\hat{\mathbf{r}}$$

b) Find expressions for the time derivatives of the unit vectors, and use this information to derive the expressions for acceleration and velocity in spherical polar coordinates.

(10 pts for part a); 30 pts for part b))

See solutions posted under "solutions for spherical polar coordinates".

2. This question deals with the celestial mechanics derivations we did in class. You can use as starting points the differential equation :

$$\ddot{r} - \frac{h^2}{r^3} = \frac{-GM}{r^2}$$

along with the definitions $h = r^2\,\dot{\phi}$, and $u = 1/r$.

In class we showed how the chain rule allows us to make the transformation :

$$\frac{dr}{dt} = -h\,\frac{du}{d\phi}$$

Using this transformation for dr/dt and the other starting points, use the chain rule to express $\ddot{r}$ in terms of u and ϕ and finally show how we obtain the differential equation (15 pts for this question):

$$\frac{d^2 u}{d\phi^2} + u = \frac{G M}{h^2}$$

Solution : We wish to transform the second time derivative of r using the substitution r = 1/u : We have already shown in class that :

$$\frac{dr}{dt} = -h \frac{du}{d\phi} \equiv w$$

We define the first derivative as the variable w to help us apply the chain rule again. We can write :

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(-h \frac{du}{d\phi}\right) = \frac{dw}{dt}$$

Where we use the definition of w above. We can apply the chain rule to dw/dt to obtain :

$$\frac{dw}{dt} = \frac{dw}{d\phi} \frac{d\phi}{dt}$$

Now, since

$$w = -h \frac{du}{d\phi} \Rightarrow \frac{dw}{d\phi} = -h \frac{d^2 u}{d\phi^2}$$

and we know from our definitions that

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{h}{r^2} = h u^2$$

Therefore, we find that :

$$\frac{d^2 r}{dt^2} = \frac{dw}{dt} = \frac{dw}{d\phi} \cdot \frac{d\phi}{dt} = -h \frac{d^2 u}{d\phi^2} \cdot h u^2 = -h^2 u^2 \frac{d^2 u}{d\phi^2}$$

Now, we can transform each term in the original differential equation :

$$\frac{d^2 r}{dt^2} = -h^2 u^2 \frac{d^2 u}{d\phi^2}$$

$$\frac{h^2}{r^3} = h^2 u^3$$

$$-\frac{G\,M}{r^2} = -G\,M\,u^2$$

Combining these we get :

$$\ddot{r} - \frac{h^2}{r^3} = \frac{-G\,M}{r^2} \to -h^2\,u^2\,\frac{d^2 u}{d\phi^2} - h^2\,u^3 = -G\,M\,u^2$$

Dividing through by $-h\,u^2$ yields our final result:

$$\mathbf{\frac{d^2 u}{d\phi^2} + u = \frac{G\,M}{h^2}}$$