PHYS 301
HOMEWORK #8

Due : 30 March 2012

For problems 1 and 2, compute the complex Fourier coefficients for the indicated functions and write out the first the non zero terms of the complex series to $c_5$and c_{-5}. Use these coefficients to write out the corresponding Fourier sin-cos series. Compute the coefficients by hand, although you may use *Mathematica* to verify your solutions. For all problems, you may use symmetry arguments to facilitate the computation of coefficients, but you must explain explain how symmetry yields your results.

1. Consider the function :

$$f(x) = \begin{cases} 0, & -\pi < x < \pi/2 \\ 1, & \pi/2 < x < \pi \end{cases}$$

Solution : We begin by computing coefficients :

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{\pi/2}^{\pi} 1\,dx = \frac{1}{4}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx = \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{-inx}\,dx = \frac{-1}{2\pi i n}\,e^{-inx}\Big|_{\pi/2}^{\pi} = \frac{-1}{2\pi i n}\left(e^{-in\pi} - e^{-in\pi/2}\right)$$

$$= \frac{-1}{2\pi i n}\,(\cos(n\pi) - i\sin(n\pi) - (\cos(n\pi/2)) - i\sin(n\pi/2))$$

We know from much experience that sin (n π) = 0; grouping the other terms :

$$= \frac{-1}{2\pi i n}[\cos(n\pi) - \cos(n\pi/2) + i\sin(n\pi/2)]$$

We can write the coefficients for various values of n (remember that we also find coefficients for negative values of n :)

$$c_1 = \frac{-1}{2\pi i(1)}\,(-1 + i) = \frac{-1}{2\pi i}\,(-1 + i)$$

$$c_{-1} = \frac{-1}{2\pi i(-1)}\,(-1 - i) = \frac{-1}{2\pi i}\,(1 + i)$$

$$c_2 = \frac{-1}{2\pi i(2)}(2)$$

$$c_{-2} = \frac{-1}{2\pi i(-2)}(2) = \frac{-1}{2\pi i(2)}(-2)$$

$$c_3 = \frac{-1}{2\pi i(3)}(-1-i)$$

$$c_{-3} = \frac{-1}{2\pi i(-3)}(-1+i) = \frac{-1}{2\pi i(3)}(1-i)$$

$$c_4 = 0 = c_{-4}$$

$$c_5 = \frac{-1}{2\pi i * 5}(-1+i); \quad c_{-5} = \frac{-1}{2\pi i(-5)}(-1+i)$$

Our Fourier expansion becomes :

f (x) =

$$\frac{1}{4} - \frac{1}{2\pi i}\left\{\left[(-1+i)e^{ix} + (1+i)e^{-ix}\right] + \left[\frac{2e^{2ix} - 2e^{-2ix}}{2}\right] + \left[\frac{(-1-i)e^{3ix} + (1-i)e^{-3ix}}{3}\right] + \right\}$$

Grouping terms inside each bracket, we get f (x) =

$$\frac{1}{4} - \frac{1}{2\pi i}\left[-\left(e^{ix} - e^{-ix}\right) + i\left(e^{ix} + e^{-ix}\right) + \frac{2\left(e^{2ix} - e^{-2ix}\right)}{2} - \frac{\left(e^{3ix} - e^{-3ix}\right)}{3} - i\frac{\left(e^{3ix} + e^{-3ix}\right)}{3}\right]$$

Remembering the definitions :

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \qquad \sin nx = \frac{e^{inx} + e^{-inx}}{2i}$$

we get :

$$f(x) = \frac{1}{4} - \frac{1}{\pi}\left[-\sin x + \cos x + \frac{2\sin 2x}{2} - \frac{\sin 3x}{3} - \frac{\cos 3x}{3} + \ldots\right] =$$

$$\frac{1}{4} - \frac{1}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \ldots\right] + \frac{1}{\pi}\left[\sin x - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \ldots\right]$$

and this matches the answer for the sine cosine series given for the answer of problem 5.3 on the bottom of p. 354.

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2. Consider the function :

$$f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$ (intervals have been corrected; use these intervals for this Q)

Solution : We follow the recipe for finding the complex coefficients.

$$c_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_0^{\pi} x\,dx\right] = \frac{\pi}{2}$$

We now use the function to compute the remaining coefficients :

$$c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,e^{-i n x}\,dx + \int_0^{\pi} x\,e^{-i n x}\,dx\right]$$

These integrals are simply computed using integration by parts (remember that we are omitting the 1/2 π coefficient of the integral, so we will have to divide by that once we finish these computations) :

$$\int_{-\pi}^{0} -x\,e^{-i n x}\,dx = -\left(\left(\frac{-1}{i n}\,x\,e^{-i n x}\right|_{-\pi}^{0} - \left(\frac{-1}{i n}\right)\int_{-\pi}^{0} e^{-i n x}\,dx\right) =$$

$$-\left(\frac{-1}{i n}\,x\,e^{-i n x}\Big|_{-\pi}^{0} - \left(\frac{-1}{i n}\right)\left(\frac{-1}{i n}\right)e^{-i n x}\Big|_{-\pi}^{0}\right) = -\left[\left(0-\left(\frac{-1}{i n}(-\pi)\,e^{i n \pi}\right)\right) - \frac{1}{i^2 n^2}\left(1-e^{i n \pi}\right)\right] =$$

$$\frac{\pi\,e^{i n \pi}}{i n} + \frac{1-e^{i n \pi}}{i^2 n^2} = \frac{i n \pi\,e^{i n \pi} + 1 - e^{i n \pi}}{i^2 n^2} = \frac{e^{i n \pi}(i n \pi - 1) + 1}{i^2 n^2} = \frac{-1 + e^{i n \pi}(1 - i n \pi)}{n^2} \qquad (1)$$

For the integral between 0 and π :

$$\int_0^{\pi} x\,e^{-i n x}\,dx = \frac{-1}{i n}\,x\,e^{-i n x}\Big|_0^{\pi} - \left(\frac{-1}{i n}\right)\int_0^{\pi} e^{-i n x}\,dx = \frac{-1}{i n}\,x\,e^{-i n x}\Big|_0^{\pi} - \frac{1}{i^2 n^2}\,e^{-i n x}\Big|_0^{\pi} =$$

$$\frac{e^{-i n x}(1 + i n x)}{n^2}\Big|_0^{\pi} = \frac{e^{-i n \pi}(1 + i n \pi) - 1}{n^2} \qquad (2)$$

Remembering that $e^{i n \pi} = (-1)^n$ for integer values of n, we add the final expressions in eqs. 1 and 2 to get a final coefficient. We find that :

$$\frac{-1 + e^{i n \pi}(1 - i n \pi)}{n^2} + \frac{e^{-i n \pi}(1 + i n \pi) - 1}{n^2} = \frac{-2 + 2(-1)^n}{n^2} = \begin{cases} 0, & n \text{ even} \\ \frac{-4}{n^2}, & n \text{ odd} \end{cases}$$

Now, return to the beginning of the problem where we reminded ourselves to divide this by 2 π, so we have :

$$c_n = \begin{cases} 0, & n \text{ even} \\ \frac{-2}{n^2 \pi}, & n \text{ odd} \end{cases}$$

Thus the complex series expansion for this function is :

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[e^{ix} + e^{-ix} + \frac{e^{3ix} + e^{-3ix}}{3^2} + \frac{e^{5ix} + e^{-5ix}}{5^2} + ...\right] =$$

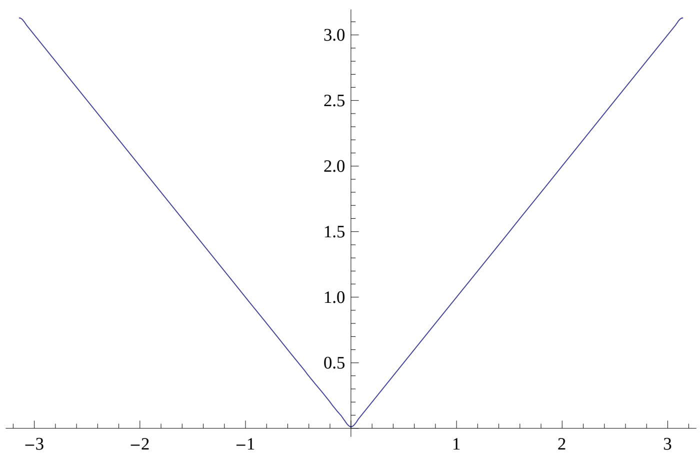
$$\frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

Constructing a plot of this expansion :

In[1174]:=

```
Plot[π / 2 - (4 / π) Sum[Cos[n x] / n^2, {n, 1, 51, 2}], {x, -π, π}]
```

Out[1174]=

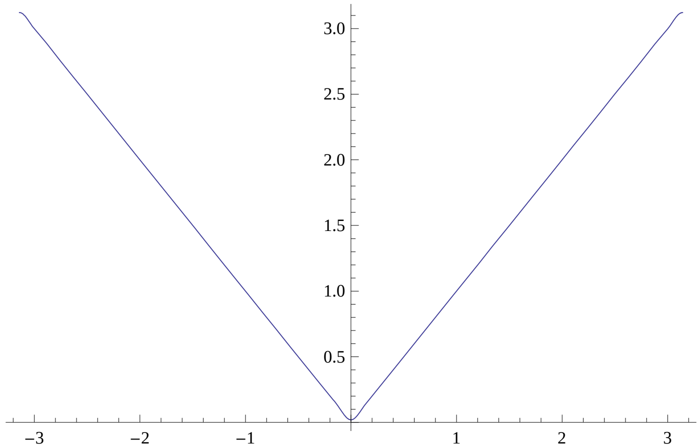


We can also plot using the complex sum :

In[1171]:=

```
Plot[π / 2 - (2 / π) Sum[Exp[ⅈ n x] / n^2, {n, -31, 31, 2}], {x, -π, π}]
```

Out[1171]=



For problems 3 - 5, you may use Mathematica to compute the definite integrals and also to determine the coefficients, but you must submit any Mathematica output you have with your assignment. Write out the first three non zero terms of each series and also plot 3 cycles of the Fourier series :

3. $f(x) = x, \; -1 < x < 1$

Solution :

In these problems, remember that the form of the expansion and integrals is different, involving sin(n π x/L) and cos(n π x/L) terms. For this problem, L = 1, and since f is odd, we expect only sin terms. Verifying :

```
a0 = 1/1 Integrate[x, {x, -1, 1}]
```

0

```
an = 1/1 Integrate[Cos[n π x/1] x, {x, -1, 1}]
```

0

```
bn = Integrate[x Sin[n π x], {x, -1, 1}]
```

$$\frac{-2\, n\, \pi \,\mathrm{Cos}[n\, \pi] + 2\, \mathrm{Sin}[n\, \pi]}{n^2\, \pi^2}$$

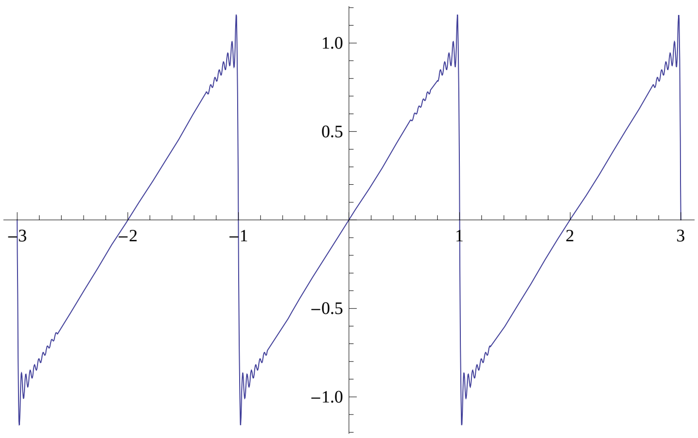
and we recognize that these coefficients reduce to :

$$\frac{-2\,(-1)^n}{n\,\pi}$$

so that :

$$f(x) = \left(\frac{2}{\pi}\right)\left[\sin(\pi x) - \frac{2\sin(2\pi x)}{2} + \frac{3\sin(3\pi x)}{3} + \ldots\right]$$

```
Plot[(-2/π) Sum[(-1)^n Sin[n π x]/n, {n, 1, 51}], {x, -3, 3}]
```



And we have 3 cycles of the function.

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4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 3 \end{cases}$

Solution : We will follow our standard procedure of finding coefficients; for this problem, we need to be mindful of how we compute these. Since the function is 2 L periodic on [-1, 3], 2 L = 4 and thus L = 2 :

$$a0 = \frac{1}{2}\int_{-1}^{3} f(x)\,dx = \frac{1}{2}\int_{0}^{3} 1\cdot dx = \frac{3}{2}$$

$$an = \frac{1}{2}\int_{0}^{3}\cos(n\pi x/2)\,dx = \frac{1}{2}\left(\frac{2}{n\pi}\right)\sin(n\pi x/2)\Big|_{0}^{3} = \frac{1}{n\pi}\sin(3n\pi/2)$$

$$bn = \frac{1}{2}\int_{0}^{3}\sin(n\pi x/2)\,dx = \frac{1}{2}\left(\frac{-2}{n\pi}\right)\cos(n\pi x/2)\Big|_{0}^{3} = \frac{1}{n\pi}(1 - \cos(3n\pi/2)$$

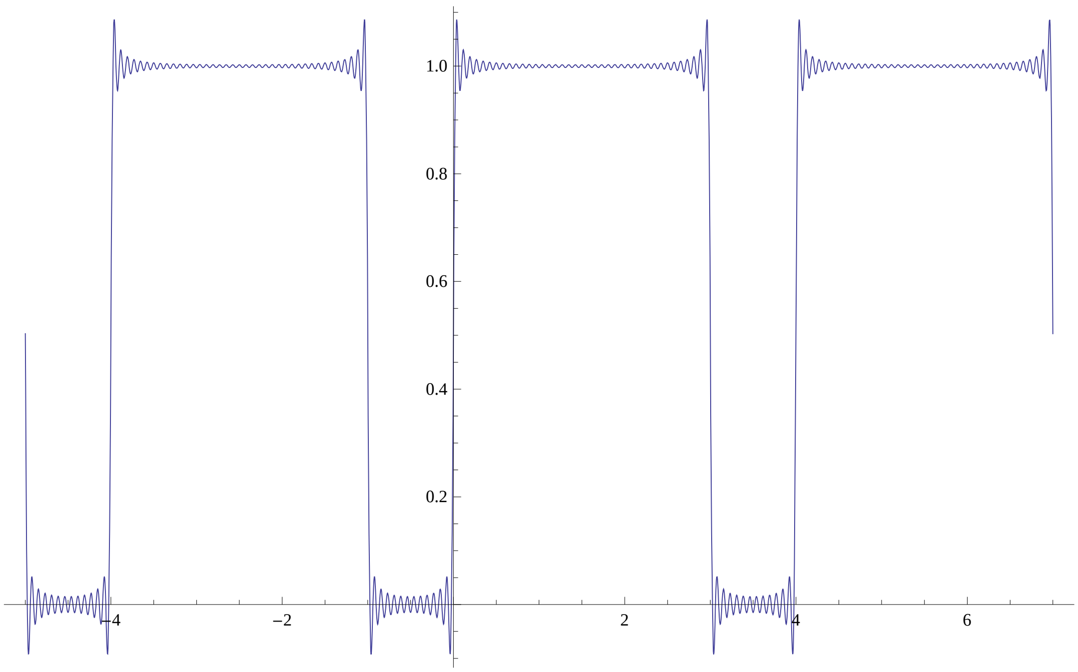
Evaluating the coefficients for different values of n, we obtain :

$$f(x) = \frac{3}{4} + \frac{1}{\pi}\left[-\cos(\pi x/2) + \frac{\cos(3\pi x/2)}{3} - \frac{\cos(5\pi x/2)}{5} + \ldots\right] +$$

$$\left(\frac{1}{\pi}\right)\left[\sin(\pi x/2) + \frac{2\sin(2\pi x/2)}{2} + \frac{\sin(3\pi x/2)}{3} + \ldots\right]$$

Using the coefficients we would get from the Mathematica calculation :

```
Plot[3 / 4 + Sum[Sin[3 n π / 2] Cos[n π x / 2] / (n π) +
    2 Sin[3 n π / 4] ^2 Sin[n π x / 2] / (n π), {n, 1, 51}], {x, -5, 7}]
```



5. Problem 9 - 23 from Boas on p. 371

Solution : The first step in the solution is to write the function representing the string. Between 0 and L/2, the string is represented by the straight line of intercept 0 and slope h/(L/2), therefore, the first half of the line is :

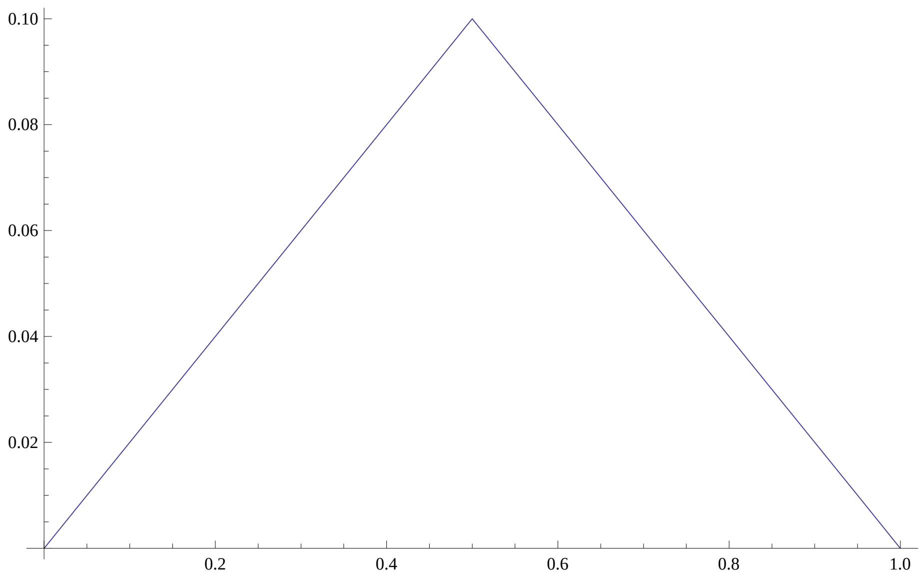
$$f(x) = (2h/L)x, \; 0 < x < L/2$$

The descending side has slope of - (2 h/L) and intercept 2 h, so the total function can be defined by :

$$f(x) = \begin{cases} (2h/L)x, & 0 < x < L/2 \\ -(2h/L)x + 2h, & L/2 < x < L \end{cases}$$

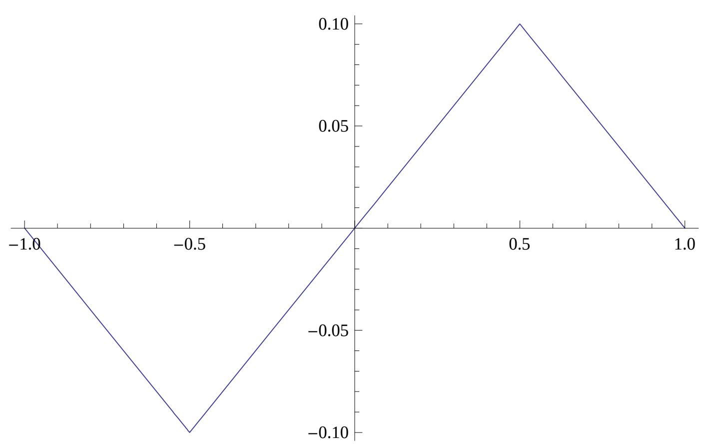
We can plot this function to verify it reproduces the string as described. We will need to define specific values for h and L since Mathematica cannot plot undefined symbols;
setting h = 0.1 and L = 1 :

```
Clear[f, h, L]
h = 0.1; L = 1.0;
f[x_] := Which[0 < x < L / 2, (2 h / L) x, L / 2 < x < L, 2 h - (2 h / L) x];
Plot[f[x], {x, 0, L}]
```



Note that the axes are not drawn to the same scale, making the perturbation in the string seem larger. The problem tells us that we want to consider the sin series generated by this perturbation; this means we make the odd extension of this function from - L to 0 :

```
Clear[g]
g[x_] := Which[-L < x < -L / 2, -2 h - (2 h / L) x,
  -L / 2 < x < L / 2, (2 h / L) x, L / 2 < x < L, 2 h - (2 h / L) x]
Plot[g[x], {x, -L, L}]
```



Now we compute Fourier coefficients for the function g (x) that is 2 L periodic on [-L, L]. Since this is an odd function, we know that a0 = an = 0; finding the bn coefficients :

$$b_n = \frac{1}{L} \int_{-L}^{L} g(x) \,\mathrm{Sin}(n \pi x / L)\, dx = \frac{2}{L} \int_{0}^{L} g(x) \,\mathrm{Sin}(n \pi x / L)\, dx$$

Using Mathematica to solve this definite integral :

```
Clear[h, L]
Do[Print["for n = ", k, "+4k,  the coefficient = ", (2 / L)
   (Integrate[(2 h / L) x Sin[n π x / L], {x, 0, L / 2}, Assumptions → Mod[n, 4] == k] + Integrate[
      (2 h - (2 h / L) x) Sin[n π x / L], {x, L / 2, L}, Assumptions → Mod[n, 8] == k])], {k, 0, 3}]
```

for n = 0+4k, the coefficient = 0

for n = 1+4k, the coefficient = $\frac{8\,h}{n^2\,\pi^2}$

for n = 2+4k, the coefficient = 0

for n = 3+4k, the coefficient = $-\frac{8\,h}{n^2\,\pi^2}$

These results allow us to write the Fourier series :

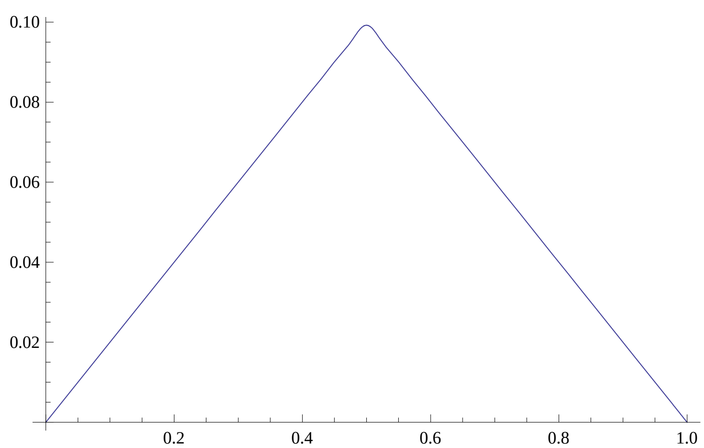
$$f(x, 0) = \frac{8\,h}{\pi^2}\left[\sin(\pi\,x/L) - \frac{\sin(3\,\pi\,x/L)}{3^2} + \frac{\sin(5\,\pi\,x/L)}{5^2} - \frac{\sin(7\,\pi\,x/L)}{7^2} + \ldots\right] =$$

$$\frac{8\,h}{\pi^2}\sum_{n,\,odd}^{\infty}(-1)^{n+1}\,\frac{\sin(n\,\pi\,x/L)}{n^2}$$

Providing again values for h and L to plot this series :

```
In[1175]:= h = 0.1; L = 1.0;
Plot[(8 h / π^2) Sum[(i) ^ (n + 3) Sin[n π x / L] / n^2, {n, 1, 51, 2}], {x, 0, 1}]
```

Out[1176]=



Notice how the sum instructs the series to alternate the sign of every other odd term.

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6. Mathematica Assignment :

Solution: Consider the differential equation :

$$\frac{dy}{dt} = \cos(y\,t) + 1 \text{ with } y(0) = 1$$

Write a short Mathematica program and use Euler' s Method to find a numerical solution for y (t) in this equation and plot your solution for y (t) from t = 0 to to = 10 s. You must use Euler' s method and discretization techniques.

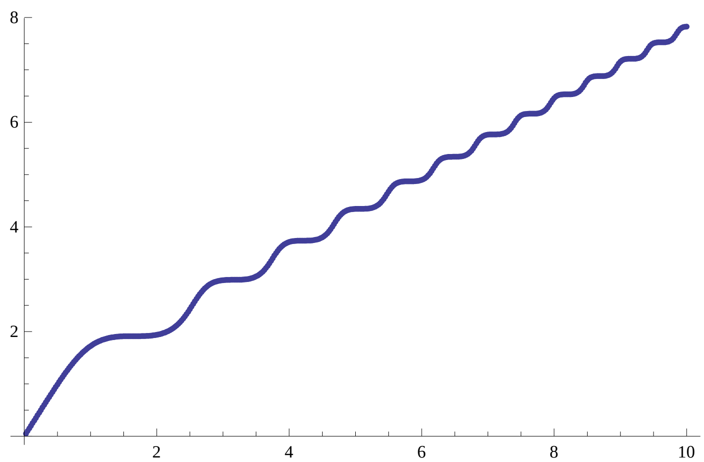
Submit your solution and plot as a .nb file electronically so I can compile your program and verify results. Answer the following questions about your solution either in the .nb file or with the rest of your homework. Show complete work and/or provide complete explanations for your answers :

What is the minimum value of the derivative? For what values of y and t does the function decrease? Why does the frequency oscillatory behavior of the solution increase as t increases?

Grading on Question 6: 10 pts for the program and 10 pts for the written explanations.

In[1263]:=
```
Clear[y, t, f]
y[0] = 0; t[0] = 0; h = 0.025;
t[n_] := t[n] = t[n - 1] + h
f[y_, t_] := Cos[y t] + 1
y[n_] := y[n] = y[n - 1] + h f[y[n - 1], t[n - 1]]
ListPlot[Table[{n h, y[n]}, {n, 1, 400}]]
```

Out[1268]=



The derivative is never negative; no matter how large y or t become, the value of the cos is bound by - 1 and 1, so the minimum value of the cos[y t] term is - 1. When this results, the total derivative becomes zero. This means the function never decreases as t increases, since the derivative is never negative. The function shows the oscillatory behavior because the function changes according to the cos (y t). Since t increases linearly in time, the frequency of the oscillations in y also increase.