PHYS 301
HOMEWORK #9
Solutions

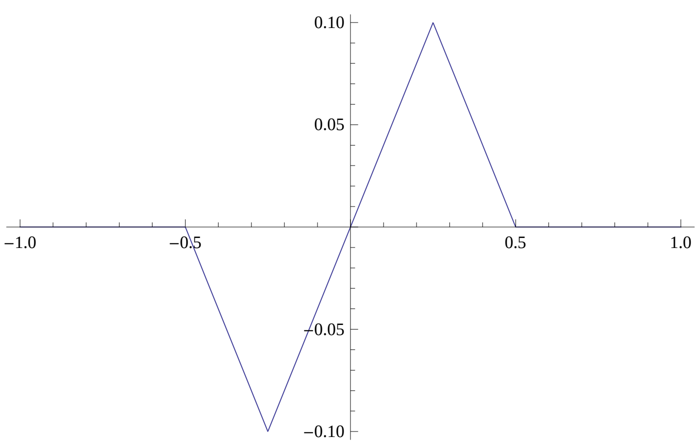
1. Boas 9 - 24 p. 371 Boas.

Solution : We are asked to find the Fourier sine series for the function defined by :

$$f(x) = \begin{cases} 4hx/L, & -L/4 < x < L/4 \\ 2h - 4hx/L, & L/4 < x < L/2 \\ 0, & L/2 < x < L \\ -2h - 4hx/L, & -L/2 < x < -L/2 \\ 0, & -L < x < -L/2 \end{cases}$$

To see what this function looks like, let' s set h = 0.1 and L = 1 :

```
h = 0.1; L = 1;
f[x_] := Piecewise[{{0, -L < x < -L / 2}, {-2 h - 4 h x / L, -L / 2 < x < -L / 4},
   {4 h x / L, -L / 4 < x < L / 4}, {2 h - 4 h x / L, L / 4 < x < L / 2}, {0, L / 2 < x < L}}]
Plot[f[x], {x, -L, L}]
```



Remember, since we are asked to find the Fourier sine series, we must make the odd extension into the left half plane. Since we know the function is odd, we can use symmetry arguments to set a_0 and all the a_n =0. Using the symmetry of odd functions to compute the b_n coefficients:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Using Mathematica :

```
Clear[h, L]
Simplify[(2 / L) (Integrate[( 4 x h / L) Sin[n π x / L], {x, 0, L / 4} ] +
    Integrate[(2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L / 2}])]
```

$$\frac{64\,h\,\mathrm{Cos}\left[\frac{n\pi}{8}\right]\,\mathrm{Sin}\left[\frac{n\pi}{8}\right]^3}{n^2\,\pi^2}$$

To find the values of the first 8 coefficients :

```
Do[Print[Simplify[(2 / L) (Integrate[( 4 x h / L) Sin[n π x / L], {x, 0, L / 4}] +
      Integrate[(2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L / 2}])]], {n, 1, 8}]
```

$$\frac{8\left(-1+\sqrt{2}\right)h}{\pi^2}$$

$$\frac{4\,h}{\pi^2}$$

$$\frac{8\left(1+\sqrt{2}\right)h}{9\,\pi^2}$$

$$0$$

$$-\frac{8\left(1+\sqrt{2}\right)h}{25\,\pi^2}$$

$$-\frac{4\,h}{9\,\pi^2}$$

$$-\frac{8\left(-1+\sqrt{2}\right)h}{49\,\pi^2}$$

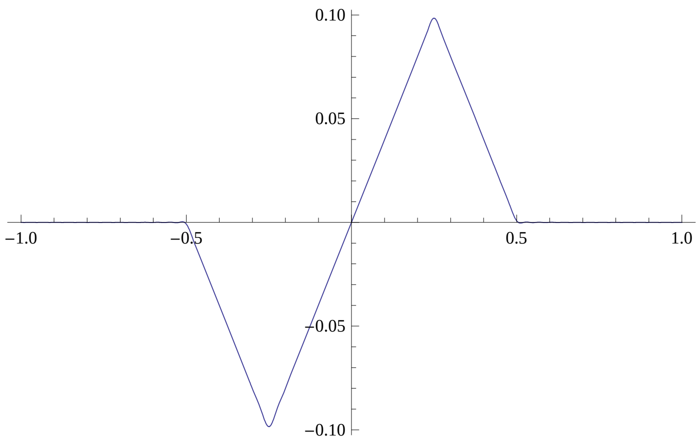
$$0$$

And our Fourier series can be written :

$$f\,(x)\;=\frac{4\,h}{\pi^2}\left[2\left(-1+\sqrt{2}\right)\sin\left(\frac{\pi\,x}{L}\right)+\frac{4\sin\left(\frac{2\pi\,x}{L}\right)}{2^2}+\frac{2\left(1+\sqrt{2}\right)\sin\left(\frac{3\pi\,x}{L}\right)}{3^2}-\frac{2\left(1+\sqrt{2}\right)\sin\left(\frac{5\pi\,x}{L}\right)}{5^2}-\frac{16\sin\left(\frac{6\pi\,x}{L}\right)}{6^2}-\frac{8\left(-1+\sqrt{2}\right)\sin\left(\frac{7\pi\,x}{L}\right)}{7^2}+\ldots\right]$$

Using the analytic form of the coefficients from above, and again providing numerically specific values for h and L, we plot the first 51 terms of this series :

```
h = 0.1; L = 1;
Plot[(64 h / π^2) Sum[Cos[n π / 8] Sin[n π / 8]^3 Sin[n π x / L] / n^2, {n, 1, 51}], {x, -L, L}]
```



For problems 2 - 4, determine the recursion relation for each differential equation and write the first three non-zero terms of the solution. If there are two branches of the solution, (as in an a_0 and an a_1 branch) write the first three non-zero terms of each branch. You may use *Mathematica* to verify solutions, but you are required to show all your work in determining the recursion relation for each equation.

Solutions :

For the next three problems, we will make repeated use of the following expressions common to power series solutions :

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{(Eq. A)}$$

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n \quad \text{(Eq. B)}$$

$$y'' = \sum_{n=2}^{\infty} n (n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n \quad \text{(Eq. C)}$$

2. $y'' + x\, y = 0$

Solution : Substituting the expressions from above into the differential equation :

$$\sum_{n=2}^{\infty} n (n-1)\, a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=2}^{\infty} n (n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Setting k = n - 2 in the first summation and k = n + 1 in the second, yields :

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0$$

Both summations now express x to the same exponent; we "strip out" the n = 0 term in the first sum to get both sums to the same lower limit :

$$2\, a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)\, a_{n+2} + a_{n-1}]\, x^n = 0$$

Now we equate terms on the left with their similar term on the right; there is no a_2 term on the right, therefore we can write $a_2 = 0$. The expression in brackets must equal zero, so we obtain our recursion relation:

$$(n+2)(n+1)\, a_{n+2} + a_{n-1} = 0 \Rightarrow \mathbf{a_{n+2} = \frac{-\, a_{n-1}}{(n+2)(n+1)}}$$

Now, we find coefficients :

$$n = 1 \Rightarrow a_3 = \frac{-a_0}{3 \cdot 2}$$

$$n = 2 \Rightarrow a_4 = \frac{-a_1}{4 \cdot 3}$$

$$n = 3 \Rightarrow a_5 = \frac{-a_2}{5 \cdot 4} = 0 \text{ since we already know that } a_2 = 0$$

$$n = 4 \Rightarrow a_6 = \frac{-a_3}{6 \cdot 5} = \frac{+a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$n = 5 \Rightarrow a_7 = \frac{-\, a_4}{7.6} = \frac{+a_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

By now, the pattern should be clear; the n^{th} coefficient is related to the $(n-3)^{\text{th}}$coefficient, so this differential equation has three separate branches. However, all the coefficients of one of those branches (a_2, a_5, a_8, ...) are zero, so we can write our complete solution:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$\mathbf{y = a_0 \left(1 - \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} - \ldots\right) + a_1 \left(x - \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} - \ldots\right)}$$

3. $y'' - 2xy' - 2y = 0$

Making use of Equations A - C we have :

$$\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} - 2x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - 2\sum_{n=0}^{\infty} a_n\,x^n = 0$$

Multiplying by x in the second sum, and re - indexing the first sum, we get :

$$\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,x^n - 2\sum_{n=1}^{\infty} n\,a_n\,x^n - 2\sum_{n=0}^{\infty} a_n\,x^n = 0$$

Stripping out the n = 0 terms in the first and third sums :

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)\,a_{n+2} - 2n\,a_n - 2a_n\right] = 0$$

The "stripped out" terms inform us that $a_2 - a_0 = 0 \Rightarrow a_2 = a_0$.
We find the recursion relation from:

$$a_{n+2} = \frac{2(n+1)\,a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2}$$

Finding coefficients :

$$a_2 = a_0;\ a_4 = \frac{a_2}{2} = \frac{a_0}{2};\ a_6 = \frac{a_4}{3} = \frac{a_0}{6}$$

$$a_3 = \frac{2a_1}{3};\ a_5 = \frac{2a_3}{5} = \frac{4a_1}{15}$$

And the series solution is :

$$\mathbf{y = a_0\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + ...\right) + a_1\left(x + \frac{2x^3}{3} + \frac{4x^5}{15} + ...\right)}$$

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4. $y'' - x^2 y' - y = 0$

Substituting the trial solution :

$$\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} - \sum_{n=1}^{\infty} n\,a_n\,x^{n+1} - \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Re - indexing :

$$\sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^n - \sum_{n=2}^{\infty}(n-1)\,a_{n-1}\,x^n - \sum_{n=0}^{\infty}a_n\,x^n = 0$$

Stripping out the n = 0 and n = 1 terms in the first and last summations :

$$2\,a_2 + 6\,a_3\,x - (a_0 + a_1\,x) + \sum_{n=2}^{\infty}[(n+2)(n+1)\,a_{n+2} - (n-1)\,a_{n-1} - a_n]\,x^n = 0$$

The stripped out terms yield :

$$2\,a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$6\,a_3 - a_1 = 0 \Rightarrow a_3 = \frac{a_1}{6}$$

The recursion relation is then :

$$(n+2)(n+1)\,a_{n+2} - (n-1)\,a_{n-1} - a_n = 0 \Rightarrow$$

$$\mathbf{a_{n+2} = \frac{a_n + (n-1)\,a_{n-1}}{(n+2)(n+1)}}$$

Starting with n = 1, we obtain :

$a_3 = \dfrac{a_1}{6}$ which is consistent with our information from the stripped out terms.

$$n = 2 \Rightarrow a_4 = \frac{a_2 + (2-1)\,a_1}{4\cdot 3} = \frac{a_0}{24} + \frac{a_1}{12} \quad \text{(remember } a_2 = a_0/2\text{)}$$

$$n = 3 \Rightarrow a_5 = \frac{a_3 + (3-1)\,a_2}{5\cdot 4} = \frac{a_3}{20} + \frac{a_2}{10} = \frac{a_1}{120} + \frac{a_0}{20}$$

$$y = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \ldots$$

$$y = a_0 + a_1\,x + \frac{a_0}{2}\,x^2 + \frac{a_1}{6}\,x^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right)x^4 + \ldots$$

$$\mathbf{y = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right) + a_1\left(x + \frac{x^3}{6} + \frac{x^4}{12} + \ldots\right)}$$

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5. Consider the differential equation :

$$(y')^2 - y = x \quad \text{with the boundary conditon } y(0) = 1$$

Find the solutions to this differential equation using power series. Recall that a quadratic equation has two solutions; one is quite simple, the other more complex. For this problem, it is likely best to write your assumed solution in the explicit form :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

and find the individual coefficients by equating the two sides of the equation. Find both solutions to this equation. One solution will truncate quickly, find the other solution out to the x^5 term. You may use *Mathematica* to help you determine the values of the coefficients.

Solution :

We write our trial solution as :

$$y = a0 + a1\,x + a2\,x^2 + a3\,x^3 + a4\,x^4 + a5\,x^5 + a6\,x^6$$

Setting x = 0, we find that a0 = 1. Substituting this trial solution into the original differential equation gives us :

```
Expand[D[y, x]^2 - y]
```

$$-a0 + a1^2 - a1\,x + 4\,a1\,a2\,x - a2\,x^2 + 4\,a2^2\,x^2 + 6\,a1\,a3\,x^2 - a3\,x^3 + 12\,a2\,a3\,x^3 + 8\,a1\,a4\,x^3 + 9\,a3^2\,x^4 - a4\,x^4 + 16\,a2\,a4\,x^4 + 10\,a1\,a5\,x^4 + 24\,a3\,a4\,x^5 - a5\,x^5 + 20\,a2\,a5\,x^5 + 12\,a1\,a6\,x^5 + 16\,a4^2\,x^6 + 30\,a3\,a5\,x^6 - a6\,x^6 + 24\,a2\,a6\,x^6 + 40\,a4\,a5\,x^7 + 36\,a3\,a6\,x^7 + 25\,a5^2\,x^8 + 48\,a4\,a6\,x^8 + 60\,a5\,a6\,x^9 + 36\,a6^2\,x^{10}$$

Now, we equate the expression above to x. Equating term by term, we find:

$$-a0 + a1^2 = 0 \Rightarrow -1 + a1^2 = 0 \Rightarrow a1 = 1 \text{ or } -1.$$

These two values of a1 will define the two solutions to the differential equation. We can show very simply that the solution with a1 = -1 truncates after two terms. Equating the x term on the left to x on the right, we have :

$$-a1\,x + 4\,a1\,a2\,x = x \Rightarrow -(-1) + 4\,(-1)\,a2 = 1 \Rightarrow a2 = 0$$

Now, since all the higher order coefficients are expressed in terms of lower order coefficients, e.g. :

$$-a2 + 6\,a1\,a3 + 4\,a2^2 = 0 \Rightarrow a3 = a2\,(1 - 4\,a2) / 6\,a1 = 0$$

Similar analyses for the higher order coefficients will show those are all equal, so one solution is simply :

$$\mathbf{y = 1 - x.}$$

Now, we find the second solution which has a1 = 1. Using this value of a1 to find a2 :

$$x\,(-a1 + 4\,a1\,a2) = x \Rightarrow -1 + 4\,a2 = 1 \Rightarrow a2 = 1/2;$$

Equating x^2 terms to find a3:

$$x^2\left(-a2 + 4\,a2^2 + 6\,a1\,a3\right) = 0 \Rightarrow -1/2 + 4\,(1/2)^2 + 6\,(1)\,a3 = 0 \Rightarrow a3 = -1/12$$

Equating x^3terms to find a4:

$$-a3\,x^3 + 12\,a2\,a3\,x^3 + 8\,a1\,a4\,x^3 \;= 0$$

Using Mathematica to help with the calculations :

```
a1 = 1; a2 = 1 / 2; a3 = -1 / 12;
Solve[- a3 + 12 a2 a3 + 8 a1 a4 == 0, a4]
```

$$\left\{\left\{a4 \to \frac{5}{96}\right\}\right\}$$

Equating x^4terms to find a5:

```
a1 = 1; a2 = 1 / 2; a3 = -1 / 12;  a4 = 5 / 96;
Solve[9 a3^2 - a4 + 16 a2 a4 + 10 a1 a5 == 0, a5]
```

$$\left\{\left\{a5 \to -\frac{41}{960}\right\}\right\}$$

and our solution is :

$$\mathbf{y = 1 + x + \frac{x^2}{2} - \frac{x^3}{12} + \frac{5\,x^4}{96} - \frac{41\,x^5}{960} + \ldots}$$

Now, I will show how we can use Mathematica to compute higher order coefficients quickly. First we write our trial solution as the function f[x] :

```
Clear[f, c, a, x]
f[x_] := Sum[a[m] x^m, {m, 0, 14}]
(*  We will make use of the fact that we know a0 =
 1 and a1 =1 for the non trival solution. *)
a[0] = 1; a[1] = 1;
c = CoefficientList[Collect[D[f[x], x]^2 - f[x], x], x];
Solve[{c[[2]] == 1, c[[3]] == 0, c[[4]] == 0, c[[5]] == 0, c[[6]] == 0,
  c[[7]] == 0, c[[8]] == 0, c[[9]] == 0, c[[10]] == 0, c[[11]] == 0, c[[12]] == 0},
 {a[2], a[3], a[4], a[5], a[6], a[7], a[8], a[9], a[10], a[11], a[12]}]
```

$$\left\{\left\{a[2] \to \frac{1}{2},\ a[3] \to -\frac{1}{12},\ a[4] \to \frac{5}{96},\ a[5] \to -\frac{41}{960},\right.\right.$$
$$a[6] \to \frac{469}{11\,520},\ a[7] \to -\frac{6889}{161\,280},\ a[8] \to \frac{24\,721}{516\,096},\ a[9] \to -\frac{2\,620\,169}{46\,448\,640},$$
$$\left.\left. a[10] \to \frac{64\,074\,901}{928\,972\,800},\ a[11] \to -\frac{1\,775\,623\,081}{20\,437\,401\,600},\ a[12] \to \frac{1\,571\,135\,527}{14\,014\,218\,240}\right\}\right\}$$

And we see that the coefficients (through a5) check our first calculation. We can also use this program to verify the other solution to the differential equation :

```
Clear[f, c, a, x]
f[x_] := Sum[a[m] x^m, {m, 0, 14}]
(*  We will make use of the fact that we know a0 =
 1 and a1 =1 for the non trival solution. *)
a[0] = 1; a[1] = -1;
c = CoefficientList[Collect[D[f[x], x]^2 - f[x], x], x];
Solve[{c[[2]] == 1, c[[3]] == 0, c[[4]] == 0, c[[5]] == 0, c[[6]] == 0,
  c[[7]] == 0, c[[8]] == 0, c[[9]] == 0, c[[10]] == 0, c[[11]] == 0, c[[12]] == 0},
 {a[2], a[3], a[4], a[5], a[6], a[7], a[8], a[9], a[10], a[11], a[12]}]
```

```
{{a[2] → 0, a[3] → 0, a[4] → 0, a[5] → 0, a[6] → 0,
  a[7] → 0, a[8] → 0, a[9] → 0, a[10] → 0, a[11] → 0, a[12] → 0}}
```

When we set a[1] = -1, we find that all the higher order coefficients are zero as before.