# PHYS 301
# HOMEWORK #10
## Solutions

1. Starting with the Legendre differential equation :

$$\left(1-x^2\right) y'' - 2\,x\,y' + m\,(m+1)\,y = 0$$

Make the substitution :

$$x = \cos\theta$$

and show the equation can be reframed as :

$$\frac{d^2 y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\,y}{d\,\theta} + m\,(m+1)\,y = 0$$

***Solution :*** Making the substitution x = cos $\theta$, we can transform the $\left(1-x^2\right)$ and first derivative term to obtain:

$$1 - x^2 = 1 - \cos^2\theta = \sin^2\theta \tag{1}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} \tag{2}$$

since x = cos $\theta$,

$$\frac{dx}{d\theta} = -\sin\theta \Rightarrow \frac{d\theta}{dx} = \frac{-1}{\sin\theta} \tag{3}$$

therefore :

$$\frac{dy}{dx} = \frac{-1}{\sin\theta}\frac{dy}{d\theta} \equiv u \tag{4}$$

Now, we wish to transform the second derivative term; using the chain rule again we have :

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{du}{dx} \tag{5}$$

$$\frac{du}{dx} = \frac{du}{d\theta}\frac{d\theta}{dx} \tag{6}$$

We already know that :

$$\frac{d\theta}{dx} = \frac{-1}{\sin\theta}$$

and $du/d\theta$ is :

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{-1}{\sin\theta}\frac{dy}{d\theta}\right) = \frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} - \frac{(-1)\,1\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} = \frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta} \tag{7}$$

Combining eqs. (3) and (7) gives :

$$\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{d\theta}\frac{d\theta}{dx} = \left[\frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}\right]\left(\frac{-1}{\sin\theta}\right) \tag{8}$$

Substituting eqs. (1), (4) and (8) into the original differential equation yields :

$$\left(1-x^2\right)y'' - 2xy' + m(m+1)y = \sin^2\theta\left[\frac{-1}{\sin\theta}\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}\right]\left(\frac{-1}{\sin\theta}\right) - 2\cos\theta\left(\frac{-1}{\sin\theta}\right)\frac{dy}{d\theta} + m(m+1)y \tag{9}$$

A little algebra brings us to :

$$\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + 2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + m(m+1)y = \boldsymbol{\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + m(m+1)y} \tag{10}$$

And we are done.

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2. The generating function for Hermite polynomials is :

$$g(x,t) = \text{Exp}\left(2xt - t^2\right) = \sum_{n=0}^{\infty}\frac{H_n(x)\,t^n}{n!}$$

where the $H_n(x)$ are the Hermite polynomials.Show that this generating function leads to the following recurrence relations :

$$H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)$$

and :

$$H_n'(x) = 2n\,H_{n-1}(x) \quad (20\text{ pts})$$

***Solutions*** : To prove the first relation, we begin by taking partial derivatives of both sides of the equation with respect to t :

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t} e^{\left(2\,xt-t^2\right)} = (2\,x - 2\,t)\, e^{\left(2\,xt-t^2\right)} = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} = \sum_{n=1}^{\infty} \frac{n\, H_n(x)\, t^{n-1}}{n!} \tag{11}$$

Now, recalling that :

$$e^{\left(2\,xt-t^2\right)} = \sum_{n=0}^{\infty} H_n(x)\, t^n\,, \tag{12}$$

we can rewrite eq. (11) as:

$$2\,(x-t)\, e^{\left(2\,xt-t^2\right)} = 2\,x \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} - 2 \sum_{n=0}^{\infty} \frac{H_n(x)\, t^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{n\, H_n(x)\, t^{n-1}}{n!} \tag{13}$$

Since n/n! = 1/(n-1)!, eq. (13) becomes:

$$2\,x \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} - 2 \sum_{n=0}^{\infty} \frac{H_n(x)\, t^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{H_n(x)\, t^{n-1}}{(n-1)!} \tag{14}$$

Now, we know from our previous work in series solutions that we wish to equate the coefficients of like powers of t. Re-indexing the second sum by setting n→n-1, and the third sum by setting n→n+1, we have:

$$2\,x \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} - 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)\, t^n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{H_{n+1}(x)\, t^n}{n!} \tag{15}$$

In the second sum in eq. (15) we can write 1/(n - 1)! = n/n! and obtain :

$$2\,x \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} - 2 \sum_{n=0}^{\infty} \frac{n\, H_{n-1}(x)\, t^n}{n!} = \sum_{n=1}^{\infty} \frac{H_{n+1}(x)\, t^n}{n!}$$

Equating coefficients of like powers of t, and dividing through by n! yields the first recursion relation :

$$2\,x\, H_n(x) - 2\,n\, H_{n-1}(x) = H_{n+1}(x)$$

To show the second relationship, take the partial derivative with respect to x :

$$\frac{\partial g(x,t)}{\partial x} = \frac{\partial}{\partial x} e^{\left(2\,xt-t^2\right)} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} \Rightarrow$$

$$2\,t\, e^{\left(2\,xt-t^2\right)} = 2 \sum_{n=0}^{\infty} \frac{H_n(x)\, t^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{H'_n(x)\, t^n}{n!}$$

Re - index the first sum above by setting n -> n + 1 and recall that 1/(n - 1)! = n/n! :

$$2\sum_{n=1}^{\infty}\frac{H_{n-1}(x)\,t^n}{(n-1)!} = 2\sum_{n=0}^{\infty}\frac{n\,H_{n-1}(x)\,t^n}{n!} = \sum_{n=0}^{\infty}\frac{H'_n(x)\,t^n}{n!}$$

Equating coefficients of like powers of t and dividing out common factors of n! gives :

$$2\,n\,H_{n-1}(x) = H'_n(x)$$

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3. Expand in a Legendre series (showing the first three non zero terms) :

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x^2, & 0 < x < 1 \end{cases}$$

***Solution*** : The general form of a Legendre series is :

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x)$$

$$\text{where } c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,P_m(x)\,dx$$

We find the first several coefficients:

$$c_0 = \frac{1}{2}\int_0^1 f(x)\,P_0(x)\,dx = \frac{1}{2}\int_0^1 x^2\cdot 1\,dx = \frac{1}{6}$$

$$c_1 = \frac{3}{2}\int_0^1 x^2\cdot x\,dx = \frac{3}{8}$$

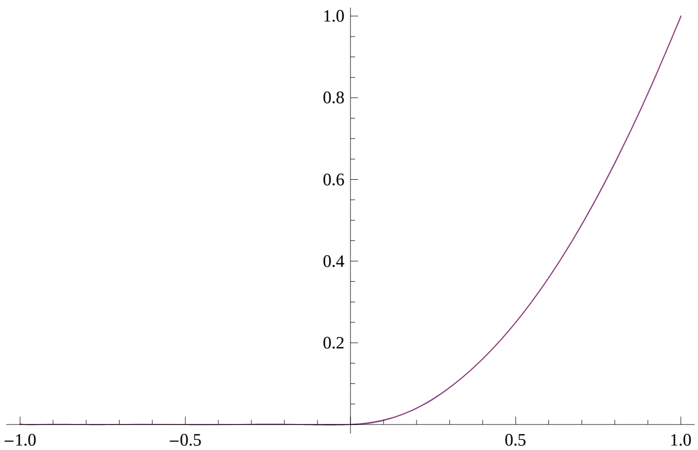
$$c_2 = \frac{5}{2}\int_0^1 x^2\cdot\frac{1}{2}\left(3x^2-1\right)dx = \frac{1}{3}$$

and our series is :

$$f(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) = \frac{1}{6} + \frac{3x}{8} + \frac{1}{3}\cdot\frac{1}{2}\left(3x^2-1\right) + \ldots$$

We write a short Mathematica program to plot the first 15 terms of the Legendre series and show its convergence to f (x) :

```
Clear[c, x, f, m]
f[x_] := Which[-1 < x < 0, 0, 0 < x < 1, x^2]
c[m_] := c[m] = ((2 m + 1 ) / 2) Integrate[f[x] LegendreP[m, x], {x, -1, 1}]
Plot[{f[z], Sum[c[m] LegendreP[m, z], {m, 0, 14}]}, {z, -1, 1}]
```

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4. Expand in a Legendre series (showing the first three non zero terms) :

$$f(x) = \cos x \quad -1 < x < 1$$

Solution : First we note that arc tan x is an odd function (prove this to yourself either by graphing it or finding the Taylor expansion). This means that we know the Legendre series will consist only of odd terms, and we calculate the first three odd coefficients :

```
Do[ Print["c", n, " = ",
  ((2 n + 1) / 2) Integrate[LegendreP[n, x] Cos[x], {x, -1, 1}]], {n, 0, 5}]
```

c0 = Sin[1]

c1 = 0

c2 = $\frac{5}{2}$ (6 Cos[1] - 4 Sin[1])

c3 = 0

c4 = $\frac{9}{2}$ (-190 Cos[1] + 122 Sin[1])

c5 = 0

and along the way verify that the cos x is even; with these coefficients we have :

$f(x) = c_0 P_0(x) + c_2 P_2(x) + c_4 P_4(x) + \ldots$

We write a short Mathematica program to write the first 12 terms of the Legendre series and plot it along with a graph of cos z on the same set of axes: :

```
Clear[c, f, x]
f[x_] := Cos[x]
c[m_] := c[m] = ((2 m + 1) / 2) Integrate[f[x] LegendreP[m, x] // N, {x, -1, 1}]

Plot[{f[z], Sum[c[p] LegendreP[p, z], {p, 0, 11}]}, {z, -1, 1}]
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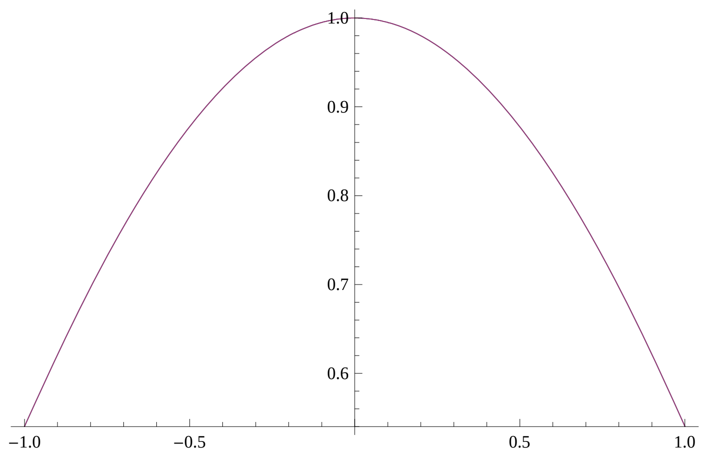


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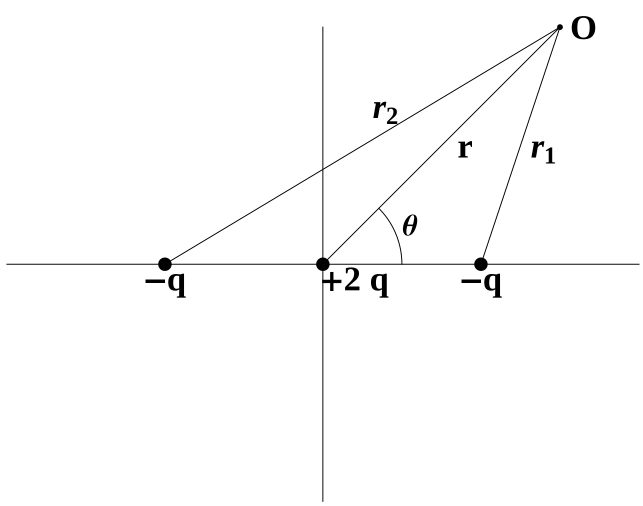
5. Consider three charges lying along the x axis. A charge of - q is at (d, 0), a charge of 2 q is at the origin, and a charge of - q lies at (-d, 0). Use Legendre polynomials to determine the potential due to this arrangement.




***Solution :*** We find the total potential at the point O by summing the individual potentials due to the charges (the principle of superposition). The potential due to the $i^{\text{th}}$ charge is:

$$V_i = \frac{k\, q_i}{r_i}$$

The potential due to the charge at the origin is easily expressible as :

$$V_0 = \frac{2\,k\,q}{r}$$

To express the potential due to the charge at (d,0), we use the law of cosines and write $r_1$ as:

$$r_1{}^2 = r^2 + d^2 - 2\,r\,d\cos\theta \Rightarrow r_1 = r\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}$$

thus the potential due to charge 1 becomes :

$$V_1 = -\frac{k\,q}{r}\,\frac{1}{\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}}$$

If we look carefully at the radical, we see that this is just the generating function for Legendre Polynomials where (d/r) takes the place of h and cos $\theta$ represents x. Thus, we can write this potential as :

$$V_1 = -\frac{k\,q}{r}\,\frac{1}{\sqrt{1 + (d/r)^2 - 2\,(d/r)\cos\theta}} = \sum_{m=0}^{\infty} P_m(\cos\theta)\,(d/r)^m$$

For the charge at (-d, 0), we first express the distance $r_2$ in terms of r, d and $\theta$. It is important to remember here that the angle between the two lines opposite $r_2$ is now 180-$\theta$, so the law of cosines gives us:

$$r_2{}^2 = r^2 + d^2 - 2\,r\,d\cos(180 - \theta) = r^2 + d^2 + 2\,r\cos\theta \Rightarrow r_2 = r\sqrt{1 + (d/r)^2 + 2\,r\,d\cos\theta}$$

and that the potential is expressed as :

$$V_2 = -\frac{k\,q}{r}\cdot\frac{1}{\sqrt{1 + (d/r)^2 + 2\,(d/r)\cos\theta}}$$

Notice that the expression for $V_2$ differs from $V_1$ in the sign of the 2 (d/r) cos $\theta$ term. This is mathematically equivalent to replacing h by -h, which allows us to express $V_2$ as:

$$V_2 = -\frac{k\,q}{r}\cdot\frac{1}{\sqrt{1 + (d/r)^2 + 2\,(d/r)\cos\theta}} = \sum_{m=0}^{\infty} P_m(\cos\theta)\,(-d/r)^m =$$

$$\sum_{m=0}^{\infty} (-1)^m\,P_m(\cos\theta)\,(d/r)^m$$

Thus, our total expression for the potential at point O is :

$$V = V_1 + V_2 + V_0 = \frac{-k\,q}{r}\left[\sum_{m=0}^{\infty} P_m(\cos\theta)\,(d/r)^m + \sum_{m=0}^{\infty}(-1)^m P_m(\cos\theta)\,(d/r)^m\right] + 2\,\frac{k\,q}{r}$$

The expression for potential involves two summations; notice that the sums add if m is even, but will cancel if m is odd, therefore we can write the total potential as :

$$V = \frac{2\,k\,q}{r}\left[1 - \sum_{m=0,\,\text{even}}^{\infty} P_m(\cos\theta)\,(d/r)^m\right]$$

Writing the first few terms explicitly gives :

$$V = \frac{2\,k\,q}{r}\left[1 - \left(1 + P_2(\cos\theta)\,(d/r)^2 + P_4(\cos\theta)\,(d/r)^4 + ...\right)\right] =$$

$$-\frac{2\,k\,q}{r}\left[\frac{1}{2}\left(3\cos^2\theta - 1\right)(d/r)^2 + \frac{1}{8}\left(3 - 30\cos^2\theta + 35\cos^4\theta\right)(d/r)^4\right]$$

If r >> d, you can see how quickly this expression converges; if we keep just the first term, we can approximate the potential field at O as :

$$V \approx -\frac{k\,q}{r}\left[\left(3\cos^2\theta - 1\right)(d/r)^2\right] = -\frac{k\,q\,d^2}{r^3}\left(3\cos^2\theta - 1\right)$$

Recalling informaton from the first part of the course, we know that the electric field is derived from the scalar potential, so that we can write :

$$\mathbf{E} = -\nabla V$$

Solving for the electric field using cylindrical coordinates:

$$-\nabla V = -\left(\frac{\partial}{\partial r} V\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta} V\,\hat{\boldsymbol{\theta}}\right) = k\,q\left(\frac{-3\,d^2}{r^4}\left(3\cos^2\theta - 1\right)\hat{\mathbf{r}} - \left(\frac{d^2}{r^4}\right)(6\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}}\right) =$$

$$-\,3\,k\,q\left(\frac{d^2}{r^4}\right)\left[\left(3\cos^2\theta - 1\right)\hat{\mathbf{r}} + \sin 2\,\theta\,\hat{\boldsymbol{\theta}}\right]$$

If $\theta = \pi/2$, the electric field is purely radial and at large distances from the origin, and its magnitude is the well known result:

$$3\,k\,q\left(\frac{d^2}{r^4}\right)$$