PHYS 301
HOMEWORK #11-- SOLUTIONS

1. #3 P. 626 : We have a semi - infinite plate of width π. The vertical sides are held at 0 degrees and the bottom edge has the boundary condition T (x, 0) = cos x. We are asked to find the temperature distribution throughout the plate. We know from having solved many similar problems that the general solution will be of the form :

$$T(x, y) = \Sigma B_n \sin(k x) e^{-k y}$$

The condition that T (π, y) = 0 implies that sin (k π) = 0 $\Rightarrow$ k π = n π such that k = n , and our general solution can be written as :

$$T(x, y) = \Sigma B_n \sin(n x) e^{-n y}$$

The lower edge condition implies :

$$T(x, 0) = \cos x = \Sigma B_n \sin(n x)$$

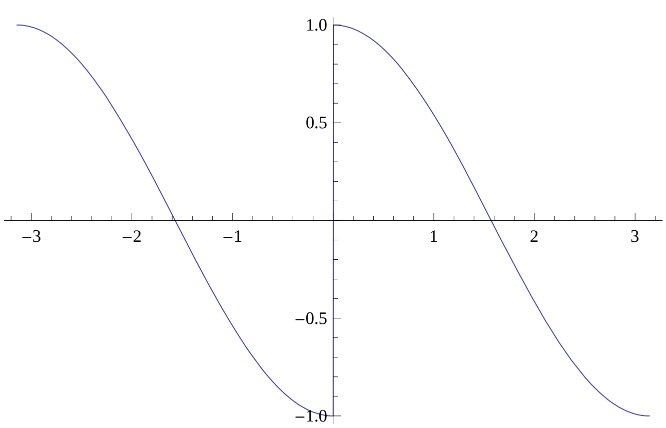
We recognize that if we can expand cos x in a Fourier sine series, we can solve for the coefficients employing the definition of the Fourier coefficients :

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(n x)\, dx$$

Be sure to understand that we had to expand cos x as an odd function on the interval (-π, π), so that the function we are considering is :

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$$

so that on (-π, π), f (x) looks like :



In this case, the Fourier b_n coefficients are equal to the B_n coefficients we need for our general solution, so we have :

$$B_n = b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(n x)\, dx = \frac{2}{\pi}\frac{n}{n^2-1}(1+\cos(n\pi))$$

When n is odd, $\cos(n\pi)$ is - 1 and the coefficients are zero; for even n, $\cos(n\pi)$ is 1 and we have :

$$b_n = \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

Finally we can use these coefficients and write our solution :

$$T(x, y) = \frac{4}{\pi}\sum_{\text{even n}}^{\infty} \frac{n}{n^2-1}\sin(n x)\, e^{-n y}$$

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2. #7 p. 627 : We know having done #3 that the solution will depend on $\sin(k x)$, and for a plate with a width of π, k = n. We know from previous work that the solution will depend on Exp[$\pm$ k y] in the vertical direction. Thus, we need to construct a boundary condition that will satisfy :

$$T(x, 1) = 0 = A e^{k y} + B e^{-k y}$$

We have seen previously (p. 624) that a solution of the form :

$$T(x, 1) = \frac{1}{2}\left(e^{k(1-y)} - e^{-k(1-y)}\right) = \sinh[k(1-y)]$$

will satisfy this condition. Setting k = n (which is dictated by the boundary condition at $x = \pi$), we have a general solution :

$$T(x, y) = \Sigma B_n \sin(n x) \sinh[n(1-y)]$$

The lower edge boundary condition implies :

$$T(x, 0) = \cos x = \Sigma B_n \sin(n x) \sinh[n]$$

Here, the Fourier b_n coefficient = B_n sinh[n], so we compute :

$$b_n = B_n \sinh(n) = \frac{2}{\pi}\int_0^{\pi}\cos x \sin(n x)\, dx$$

This is simply the integral done in question #1. Remembering that B = b/sinh (n), the complete solution becomes :

$$T(x, y) = \frac{4}{\pi}\sum_{\text{even n}}^{\infty}\frac{n}{n^2-1}\frac{\sin(n x)\sinh[n(1-y)]}{n \sinh(n)}$$

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3. #12 p. 627. For this problem, we remember that a sum of solutions to a linear differential equation is also a solution. We can break down the larger problem into two smaller ones. We will first find the temperature distribution in the square assuming only the lower edge is at 100 degrees, and then we will separately find the temperature distribution if only the left vertical edge is at 100 degrees. Consider the problem of a rectangle of length 10 cm and height of 30 cm with the bottom

edge at 100 degrees and all other sides at zero degrees. This problem was solved in the text on p. 624. The solution is equation 2.17 from the text :

$$T_1(x, y) = \frac{400}{\pi} \sum_{\text{odd } n}^{\infty} \frac{\sinh\left[\frac{n\pi}{10}(30 - y)\right] \sin\left(\frac{n\pi x}{10}\right)}{n\pi \sinh n\pi}$$

Now, if the left vertical side is heated to 100 degrees and the other three sides are at zero, our general solution will consist of a sin function in the y direction, and exponentials in the x direction. The boundary condition that T_2 (10, y) = 0 suggests the solution :

$$T_2(10, y) = \frac{1}{2} e^{k(10-x)} - \frac{1}{2} e^{-k(10-x)} = \sinh[k(10 - x)]$$

The boundary condition at y = 30, T_2 (x, 30) = 0 tells us that sin (30 k) = 0 or k = n π/30. These two pieces of information tell us that the solution will have the form :

$$T_2(x, y) = \Sigma B_n \sin(n\pi y/30) \sinh\left[\frac{n\pi}{30}(10 - x)\right] \quad (1)$$

Applying the BC that the temperature along the left vertical edge is 100 gives us :

$$T_2(0, y) = \Sigma B_n \sin(n\pi y/30) \sinh\left[\frac{n\pi}{3}\right] = 100$$

We use standard Fourier analysis to find the B_n coefficients :

$$b_n = B_n \sinh\left[\frac{n\pi}{3}\right] = \frac{2}{30}\int_0^{30} 100 \sin\left(\frac{n\pi y}{30}\right) dy = \begin{cases} 0, & n \text{ even} \\ 400/n\pi, & n \text{ odd} \end{cases}$$

Solving for the B_n coefficients and substituting into eq. (1), we find for T_2(x,y) :

$$T_2(x, y) = \frac{400}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\pi y/30) \sinh[n\pi(10 - x)/30]}{n \sinh(n\pi/3)}$$

The total solution for the temperature distribution is the sum of the two solutions, or :

$$T(x, y) = T_1(x, y) + T_2(x, y)$$

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4. #2, p. 632. This problem is similar to the first example in the book (p. 629) and first example of the heat diffusion equation done in class. For a one dimensional bar, we expect a general solution of the form :

$$u(x, t) = \Sigma B_n \sin(kx) e^{-k^2 \alpha^2 t}$$

For all times t > 0, we are told that u (0, t) = 0 and u (0, 10) = 0. The latter condition tells us that sin (10 k) = 0 ⇒ k = n π /10. The initial boundary condition (for t <= 0) is u (x, 0) = 100, so that incorporating these two results into our general solution gives us :

$$u(x, 0) = \Sigma B_n \sin(n \pi x / 10) = 100$$

We recognize immediately that the B_n are the Fourier b_n coefficients when f (x) = 100 and L = 10, so that we compute :

$$B_n = b_n = \frac{2}{10} \int_0^{10} 100 \sin(n \pi x / 10)\, dx = -\frac{200(-1 + \cos(n \pi))}{n \pi} = \begin{cases} 400 / n \pi, & \text{n odd} \\ 0, & \text{n even} \end{cases}$$

and the solution is :

$$u(x, t) = \frac{400}{\pi} \sum_{\text{odd n}}^{\infty} \frac{\sin(n \pi x / 10)\, e^{-(n \pi \alpha/10)^2 t}}{n}$$

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5. #5, p. 632. This problem differs from the previous one in that the final steady - state configuration produces temperatures different from zero. Since the sum of solutions is also a solution to the general case, we add the final result and our general solution is of the form :

$$u(x, t) = \Sigma (a_n \cos kx + b_n \sin kx)\, e^{-k^2 \alpha^2 t} + u_f = \Sigma (a_n \cos kx + b_n \sin kx)\, e^{-k^2 \alpha^2 t} + 100$$

where u_f represents the temperature distribution as t grows very large. Given that the two outer faces are held at 100^o, we expect that as t grows large, the final temperature distribution simply becomes u_f = 100. We have not yet discarded either the sin or cos solution for the spatial component of the distribution; we will use boundary conditions to determine which function to keep. If the temperature at x=0 must be 100 for all times t>0, then the summation must equal zero in order for u(0,t)=100. Since cos 0 ≠0, only the sin solution works, and our solution will involve only sin term. In order for the BC sin(10k) =0,

k=n π/2 (the total width of the solid =2), and we have:

$$u(x, t) = \Sigma B_n \sin(n \pi x / 2)\, e^{-(n \pi \alpha/2)^2 t} + 100 \qquad (2)$$

Applying the condition that at t = 0 the temperature distribution is :

$$u(x, 0) = \begin{cases} 100\, x, & 0 < x < 1 \\ 100 (2 - x), & 1 < x < 2 \end{cases}$$

Then, we can write :

$$u(x, 0) = \Sigma B_n \sin(n \pi x / 2) + 100$$

$$u(x, 0) - 100 = \Sigma B_n \sin(n \pi x / 2)$$

and we can see that the B_n coefficients are simply the Fourier coefficients for the function u (x, 0) - 100 on the interval (0, 2). We find these coefficients from :

$$B_n = b_n = \frac{2}{2}\left[\int_0^1 (100\, x - 100) \sin(n \pi / 2)\, dx + \int_1^2 -100 (x - 1)\, \mathrm{si}(n \pi / 2)\, dx\right]$$

When evaluated, these coefficients are :

$$B_n = b_n = \begin{cases} 0, & \text{n even} \\ \frac{8}{n^2\pi^2} - \frac{4}{n\pi}, & n = 1 \text{ Mod } 4 \, (n = 1, 5, 9, \ldots) \\ \frac{-8}{n^2\pi^2} - \frac{4}{n\pi}, & n = 3 \text{ Mod } 4 \, (n = 3, 7, 11, \ldots) \end{cases}$$

Substitute these values for B_n into equation (2), and we have a complete solution for th e problem.

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6. #7, p. 633

We know the general solution to the heat diffusion equation will have the form :

$$u(x, t) = \Sigma (a_n \cos nx + b_n \sin nx)\, e^{-k^2 \alpha \wedge 2\, t}$$

In this case, the ends and not just the sides are insulated, so that no energy will diffuse across the ends, meaning that we can express this boundary condition as :

$$\frac{\partial u(0, t)}{\partial x} = 0 \text{ and } \frac{\partial u(L, t)}{\partial x} = 0$$

The BC at x = 0 instructs us to discard the sin solution, since the derivative of sin (i.e., cos) is not zero at x = 0. Thus, our solution becomes :

$$u(x, t) = \Sigma a_n \cos nx\, e^{-k^2 \alpha^2 t}$$

The BC at x = L implies that $\partial/\partial x\,(\cos(k\,x)) = 0$, or that $\sin(k\,L) = 0 \Rightarrow k = n\,\pi/L$

We can use these results to write the general solution as :

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos(n\,\pi\,x/L)\, e^{-(n\,\pi\,\alpha/L)^2 t} \tag{3}$$

Now we apply the t = 0 condition that u (x, 0) = x :

$$u(x, 0) = x = \sum_{n=0}^{\infty} a_n \cos(n\,\pi\,x/L) \tag{4}$$

. Recall that the Fourier theorem tells us we can expand a function in the form :

$$f(x) = \frac{a_0}{2} + \Sigma (a_n \cos(n\,\pi\,x/L) + b_n \sin(n\,\pi\,x/L))$$

Note that the sum in eqs. (3) and (4) begin with n = 0; this is because a Fourier cos series must include the a_0 term. We can find the a_n coefficients using the well known relationships:

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^L x\,dx = L \Rightarrow \frac{a_0}{2} = \frac{L}{2}$$

$$a_n = \frac{2}{L}\int_0^L x\cos(n\,\pi\,x/L)\,dx = 2\,\frac{L}{n^2\,\pi^2}\,(-1 + \cos(n\,\pi)) = \begin{cases} 0, & \text{n even} \\ -\frac{4L}{n^2\pi^2}, & \text{n odd} \end{cases}$$

The complete solution is then :

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{\text{odd } n}^{\infty} \frac{\cos(n\pi x/L)\, e^{-(n\pi\alpha/L)^2 t}}{n^2}$$

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7. #2, p. 637 This is a standard wave equation for a string with zero initial velocity and whose ends are fixed such that $y(0, t) = y(L, t) = 0$.

Solving the wave equations yields a general solution of the form :

$$y(x, t) = \Sigma (a_k \cos kx + b_k \sin kx)(c_k \cos(k v t) + d_k \sin(k v t)).$$

The spatial boundary conditions lead us to discard the cos kx solution since cos kx cannot be zero at $x = 0$. The spatial condition at $x = L$ leads to $\sin(k L) = 0 \Rightarrow k = n\pi/L$. We are told that the initial velocity of the string is zero, this means that $\partial y(x, 0)/\partial t = 0$. This condition causes us to discard the $\sin(k v t)$ solutions, since the derivative of $\sin(k v t)$ is non - zero at $t = 0$, whereas the derivative of $\cos(k v t) = 0$ at $t = 0$. Thus, our general solution has the form :

$$y(x, t) = \Sigma b_n \sin(n\pi x/L) \cos(n\pi v t/L). \tag{5}$$

As is becoming familiar, we find the values of the coefficients by applying the boundary condition at $t = 0$ and solving for the appropriate Fourier coefficients.

The boundary condition is :

$$y(x, 0) = \begin{cases} 4hx/L, & 0 < x < L/4 \\ 2h - 4hx/L, & L/4 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

We compute the relevant Fourier coefficients from :

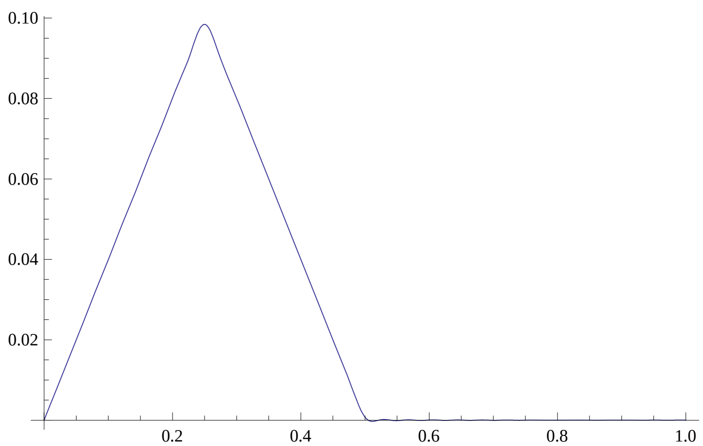
$$b_n = \frac{2}{L}\left[\int_0^{L/4} (4hx/L) \sin(n\pi x/L)\, dx + \int_{L/4}^{L/2} (2h - 4hx/L) \sin(n\pi x/L)\, dx\right]$$

Using Mathematica, you can determine these coefficients to be :

$$b_n = \frac{64h}{n^2\pi^2} \cos(n\pi/8) \sin^3(n\pi/8)$$

Substituting this expression for coefficients into equation (5) will produce the general solution for the equation. The plot below shows that these coefficients will reproduce the initial $y(x, 0)$ condition :

```
Clear[b, x, h, L]
h = 0.1; L = 1;
b[n_] := 64 h Cos[n π / 8] Sin[n π / 8]^3 / (n π)^2
Plot[Sum[b[n] Sin[n π x / L], {n, 1, 51}], {x, 0, L}]
```



We can use the Manipulate command to enable us to simulate the motion of this wave pattern :

```
Clear[b, y, x, h, L, t, v]
h = 0.1; L = 1.0; v = 1;
b[n_] := (64 h / π^2) Cos[n π / 8] Sin[n π / 8]^3 / n^2
Manipulate[Plot[Sum[b[n] Sin[n π x / L] Cos[n π v t / L], {n, 1, 31}],
  {x, 0, L}], {t, 0, 50, 0.1}]
```

Since I have to post this as a .pdf, I cannot show the interactive nature of Manipulate; you will need to type the code into an open notebook and execute it. Note that I have to provide numerical values for h, L and v in order to allow Mathematica to produce a plot. You can play around with the values; you will find that there is a trade - off between the number of terms in the sum and the speed with which Mathematica can update the simulation. Try using the "play" option in Manipulate, slowing down the simulation until you can get a sense for how the disturbance propagates down the string.

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8. #6, p .638. This is a problem in which the ends of the string are fixed at x = 0 and x = L, and the string' s initial velocity is given as the function :

$$\frac{\partial y(x, 0)}{\partial t} = \begin{cases} h, & \frac{L}{2} - w < x < \frac{L}{2} + w \\ 0, & \text{elsewhere} \end{cases}$$

The condition that the string is fixed at x = 0 and x = L implies that we use sin kx for the spatial function; the fact that the velocity is non - zero suggests we use the sin (k v t) basis functions since the derivative of sin (kvt) is not zero at t = 0, whereas the derivative of cos (k v t) is zero when t = 0. Therefore, our solution will have the general form :

$$y(x,t) = \Sigma B_n \sin(k x) \sin(k v t)$$

As we have seen (many times) before, the condition that sin (kL) = 0 ⇒ k L = n π or k = n π/L. We apply the t=0 boundary condition to solve for the b_n (remember to differentiate the sin(n π v t/L) term):

$$\frac{\partial y(x,0)}{\partial t} = \Sigma B_n (n\pi v/L) \sin(n\pi x/L) \cos(n\pi v t/L); \quad \text{when } t = 0 \Rightarrow$$

$$\frac{\partial y(x,0)}{\partial t} = \Sigma B_n (n\pi v/L) \sin(n\pi x/L)$$

In this case, the Fourier coefficients are defined as :

$$b_n = B_n (n\pi v/L) \Rightarrow B_n = \frac{b_n L}{n\pi v}$$

We compute the Fourier coefficients, using Mathematica to help calculate the terms we obtain :

$$b_n = \frac{2}{L}\int_0^L \frac{\partial y(x,0)}{\partial t} \sin(n\pi x/L)\, dx = \frac{2}{L}\int_{L/2-w}^{L/2+w} h \sin(n\pi x/L)\, dx = \frac{4h}{\pi n} \sin(n\pi/2) \sin(n\pi w/L)$$

Using these values for b_n in the definition for B_n gives us a final result :

$$y(x,t) = \frac{4hL}{\pi^2 v} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\sin(n\pi w/L)\sin(n\pi x/L)\sin(n\pi v t/L)}{n^2}$$

This solution looks quite complex. Still, note the presence of a sin (n π/2) term. This term will be zero whenever n is even (so the resulting series will contain only odd terms). Also, since sin (n π/2) alternates between 1 and - 1 for alternating odd values of n, you easily reproduce the first two terms of the series shown in the text' s answer, and determine easily the next few terms.

9, #1, p. 650, We are asked to find the temperature distribution inside a sphere of radius 1. We know the general solutions to Laplace' s equation in spherical coordinates are :

$$T(r,\theta) = \sum_{m=1}^{\infty} \left(A_m r^m + B_m r^{-(m+1)}\right) P_m(\cos\theta)$$

The constraint that we are inside the sphere requires we discard the B_m solutions, since those terms would diverge at r = 0. Then, we apply the surface condition that :

$$T(1,\theta) = \sum_{m=1}^{\infty} A_m (1)^m P_m(\cos\theta) = 35\cos^4\theta$$

This shows us that here, the coefficients A_m are just the coefficients of the Legendre series for $\cos^4\theta$. The remainder of the problem consists of computing Legendre coefficients:

$$c_m = A_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\, dx$$

We can make the substitution x = cos θ to produce the particularly simple :

$$c_m = \frac{2\,m+1}{2} \int_{-1}^{1} 35\,x^4\,P_m(x)\,dx$$

It is simple to compute these coefficients :

```
Clear[m, x, c]
c[m_] := c[m] = ((2 m + 1) / 2) Integrate[35 x^4 LegendreP[m, x], {x, -1, 1}]
Do[Print["For m = ", m, " c_m = ", c[m]], {m, 0, 6}]
```

```
For m = 0 c_m = 7
For m = 1 c_m = 0
For m = 2 c_m = 20
For m = 3 c_m = 0
For m = 4 c_m = 8
For m = 5 c_m = 0
For m = 6 c_m = 0
```

With these values of the c coefficients, we can write our solution as :

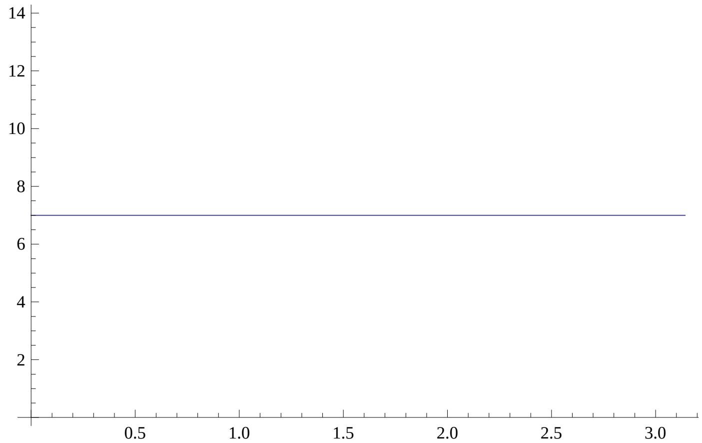
$$T(r, \theta) = \left[7\,r^0\,P_0(\cos\theta) + 20\,r^2\,P_2(\cos\theta) + 8\,r^4\,P_4(\cos\theta)\right]$$

Let' s see if this solution produces results that we expect. Since only the first term in the expansion will be non - zero at r = 0, we expect that the temperature at the center should be zero. We can show this either numerically or graphically. We will use the following code for various verification tests :

```
Clear[temp, r, θ]
temp[r_, θ_] :=
 7 + 10 r^2 (3 Cos[θ]^2 - 1) + r^4 (3 - 30 Cos[θ]^2 + 35 Cos[θ]^4)
```

Notice that I wrote out the Legendre polynomials rather than use the Mathematica LegendreP function; I did this since Mathematica will compute polynomials much faster that Legendre functions. For higher order series, I would invoke the LegendreP functions. Let' s see what a plot of temp[r, θ] will produce :

```
Plot[temp[0, θ], {θ, 0, π}]
```



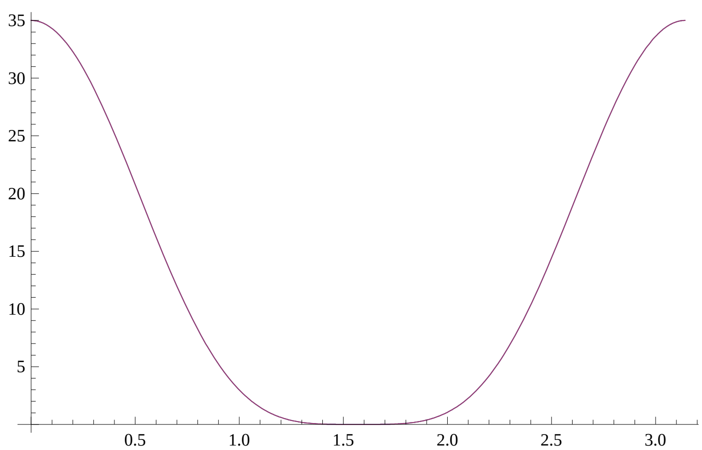
As we expect, the temperature is 7^o "everywhere" at the point at the center of the sphere. Numerically, we could have simply computed :

```
Print["The temperature at the center of the sphere = ", temp[0, 0], " degrees"]
```

```
The temperature at the center of the sphere = 7 degrees
```

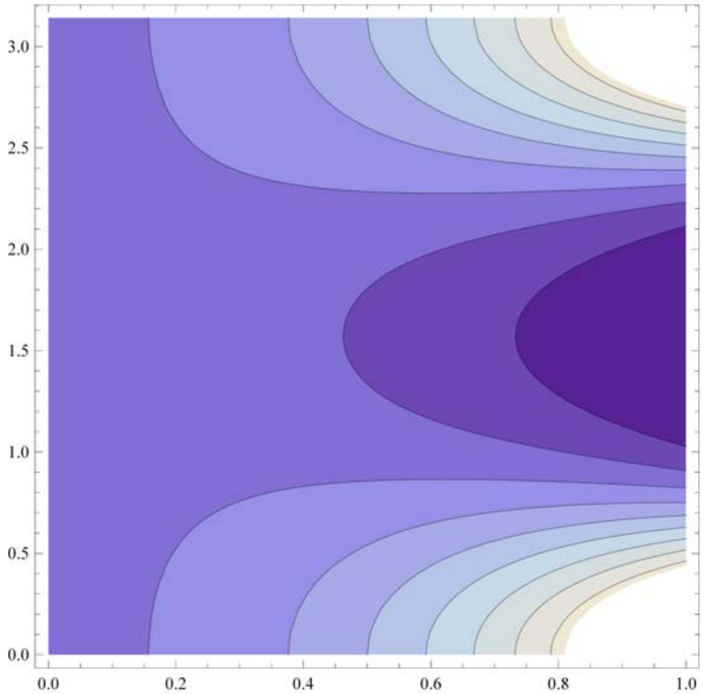
Now, let' s see what we get at the surface. I plot 35 $\cos^4 \theta$ on the same set of axes as our solution at r = 1 :

```
Plot[{35 Cos[θ]^4, temp[1, θ]}, {θ, 0, π}]
```



And our solution matches the surface boundary condition as it should. Finally, what would a contour plot of our solution look like in (r, θ) coordinates :

```
ContourPlot[temp[r, θ], {r, 0, 1}, {θ, 0, π}]
```



The radial axis lies along the abscissa, and the angle lies along the ordinate. Notice how the temperature increases if you start on the equator ($\theta = \pi/2$) and move radially in toward the center.

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10. #7, p. 650. This problem is essentially the same as the problem above except we have as our surface condition :

$$T(1, \theta) = \begin{cases} \cos\theta, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$$

The upper hemisphere is held at a temperature cos θ and the lower temperature is held at zero degrees. Making the substitution $x = \cos\theta$ this BC becomes :

$$T(1, x) = \begin{cases} x, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$$

The Legendre coefficients become :

$$c_m = \frac{2m+1}{2} \int_0^1 x\, P_m(x)\, dx$$

```
Clear[m, x, c]
c[m_] := c[m] = ((2 m + 1) / 2) Integrate[ x LegendreP[m, x], {x, 0, 1}]
Do[Print["For m = ", m, " cm = ", c[m]], {m, 0, 6}]
```

For m = 0 $c_m = \frac{1}{4}$

For m = 1 $c_m = \frac{1}{2}$

For m = 2 $c_m = \frac{5}{16}$

For m = 3 $c_m = 0$

For m = 4 $c_m = -\frac{3}{32}$

For m = 5 $c_m = 0$

For m = 6 $c_m = \frac{13}{256}$

and the temperature distribution is :

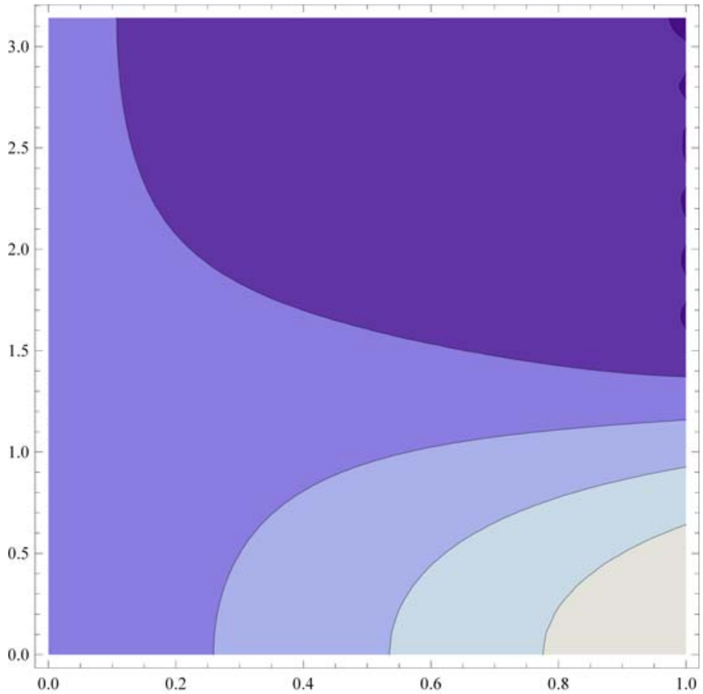
$$T(r, \theta) = \frac{1}{4} r^0 P_0(\cos\theta) + \frac{1}{2} r P_1(\cos\theta) + \frac{5}{16} r^2 P_2(\cos\theta) - \frac{3}{32} r^4 P_4(\cos\theta) + ...$$

As before, let' s write a short program to investigate the temperature distribution inside the sphere :

```
Clear[temp, c, r, θ]
c[m_] := c[m] = ((2 m + 1) / 2) Integrate[x LegendreP[m, x] // N, {x, 0, 1}]
temp[r_, θ_] := (0.25) + Sum[c[m] r^m LegendreP[m, Cos[θ]], {m, 1, 21}]
```

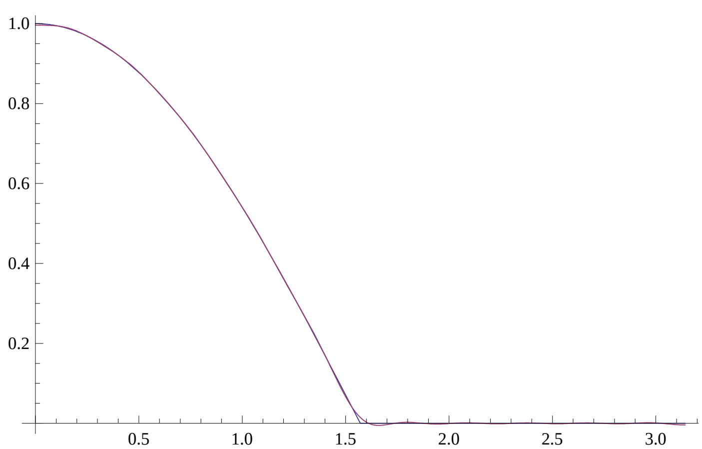
There are a few comments we can make about this program. First, notice that I start the summation at m = 1 instead of m = 0, and write the m = 0 term explicitly outside the sum. If you try to do the sum from m = 0, the first term becomes 0^0 which *Mathematica* will report as indeterminate. Second, notice that I force a numerical result in the integration. This is the work around to the “Mathematica weirdness” we discovered; otherwise *Mathematica* will produce exponentially oscillating functions for m>15 or so. The contour plot is:

```
ContourPlot[temp[r, θ], {r, 0, 1}, {θ, 0, π}]
```



To see if the solution reproduces our surface boundary condition :

```
Clear[surfacetemp, θ]
surfacetemp[θ_] := Which[π / 2 < θ < π, 0, 0 < θ < π / 2, Cos[θ]]
Plot[{surfacetemp[θ], temp[1, θ]}, {θ, 0, π}]
```



The exterior solution : Let' s go one step beyond and find the solution exterior to the sphere, and show how the exterior solution matches up with the interior solution at r = 1. Outside the sphere, the A_m coefficients must go to zero otherwise the $A_m\, r^m$ terms will go to infinity as r grows large, this gives us the solution :

$$T_{ext}(r,\ \theta) = \sum_{m=0}^{\infty} B_m\, r^{-(m+1)}\, P_m(\cos\theta)$$

The surface condition yields :

$$T_{ext}(1,\ \theta) = \sum_{m=0}^{\infty} B_m (1)^{-(m+1)} P_m(\cos\theta) = \begin{cases} \cos\theta, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$$

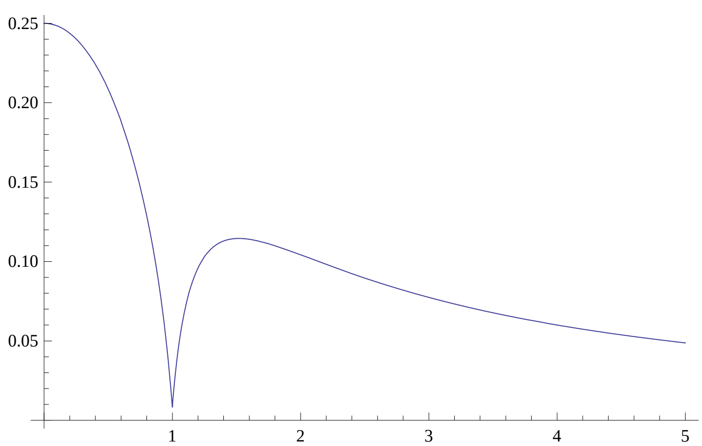
We recognize the B_m coefficients are the coefficients in the Legendre series and are the same as computed above, so our solution is :

$$T_{ext}(r,\ \theta) = \frac{1}{4} \cdot \frac{1}{r} P_0(\cos\theta) + \frac{1}{2} \cdot \frac{1}{r^2} P_1(\cos\theta) - \frac{3}{32} \cdot \frac{1}{r^4} P_3(\cos\theta) + \ldots$$

Compare the solution for T_{ext} with the interior solution; it should be clear they are the same at r = 1. Let' s compute both the interior and exterior solutions and plot the temperature distribution along the equatorial plane ($\theta = \pi/2$) :

```
Clear[tempint, tempext, c, r]
c[m_] := c[m] = ((2 m + 1) / 2) Integrate[x LegendreP[m, x] // N, {x, 0, 1}]
tempint[r_, θ_] := (0.25) + Sum[c[m] r^m LegendreP[m, Cos[θ]], {m, 1, 41}]
tempext[r_, θ_] := Sum[c[m] r^-(m+1) LegendreP[m, Cos[θ]], {m, 0, 41}]
g1 = Plot[tempint[r, π / 2], {r, 0, 1}];
g2 = Plot[tempext[r, π / 2], {r, 1, 5}];
Show[g1, g2, PlotRange → All]
```

Out[281]=



The graph above shows that the temperature is 0.25 at r = 0 as we expect; the temperature decreases as you approach the surface, reaching (almost) zero at the surface. (The discrepancy is due to the finite number of terms we use in the Legendre series). Just exterior to the sphere, the temperature increases slightly due to contributions from the upper hemisphere, but as one moves farther from the center, the temperature slowly decreases.

To get a sense of the functional dependence of T (r, $\pi/2$), I superimpose plots of the exterior solution with the curve of T = 0.25/r (the latter curve in red). Note how at larger distances, the two curves nearly coincide, suggesting that at distances r >> radius of sphere, the temperature dependence follows a 1/r law. Compare this to other situations that are described by Laplace' s equation, and think through whether this makes sense.

In[314]:=
```
g3 = Plot[tempext[r, π / 2], {r, 1, 18}];
g4 = Plot[0.25 / r, {r, 1, 18}, PlotStyle → Red];
Show[g3, g4, PlotRange → All]
```

Out[316]=