PHYS 301
HOMEWORK #1

Due : 18 Jan. 2013

All homeworks are due at the beginning of class on the day assigned. You must provide complete solutions to receive full credit. On future assignments, you will be allowed to evaluate integrals via Mathematica, but for this assignment, you must do all problems by hand.

1. Evaluate :

$$\int \sin(p\,x)\sin(q\,x)\;dx$$

where p and q are integer constants. (If you use any trig identities, you must cite them clearly or show their derivations.)

Solution : We will begin by recalling the cos addition formulae :

$$\cos(a \mp b) = \cos a \cos b \pm \sin a \sin b$$

Therefore, we can write :

$$\cos(p-q)\,x - \cos(p+q)\,x = 2\sin(p\,x)\sin(q\,x)$$

This identity allows us to write :

$$\int \sin(p\,x)\sin(q\,x)\;dx = \frac{1}{2}\int [\cos(p-q)\,x - \cos(p+q)\,x\;dx] =$$

$$\frac{1}{2}\left(\frac{\sin(p-q)\,x}{p-q} - \frac{\sin(p+q)\,x}{p+q}\right)$$

Verifying via Mathematica :

```
In[54]:= Integrate[Sin[p x] Sin[q x], x] // TraditionalForm
```

Out[54]//TraditionalForm=

$$\frac{\sin(x\,(p-q))}{2\,(p-q)} - \frac{\sin(x\,(p+q))}{2\,(p+q)}$$

---

2. Use the results of the first question to evaluate :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(m\,x)\sin(n\,x)\,dx$$

for the cases where $m = n \neq 0$; $m \neq n$; $m = n = 0$, and where m and n are integers.

Solution : Using the results of the indefinite integral above, we can write immediately :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(m\,x)\sin(n\,x)\,dx \;=\; \frac{1}{2\pi}\cdot\frac{1}{2}\left(\frac{\sin(p-q)\,x}{p-q}-\frac{\sin(p+q)\,x}{p+q}\right)\Bigg|_{-\pi}^{\pi}$$

In the case where m != n, the value of the integral is zero since the value of sin $(n\,\pi) = 0$ for any integer value of n. In the case where m = n = 0, the value of sin is zero, so the integrand is zero at all points in the interval. In the special case where m = n != 0, we get a non zero result, since the integral becomes :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(m\,x)\sin(n\,x)\,dx \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2(n\,x)\,dx \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\cos 2\,x}{2}\,dx \;=\; \frac{1}{2}$$

where we make use of the identities :

$\sin^2 x \;=\; 1 \;-\; \cos^2 x$ and $\cos 2\,x \;=\; \cos^2 x \;-\; \sin^2 x \;=\; 1-2\sin^2 x \;\Rightarrow\; \sin^2 x \;=\; (1-\cos 2\,x)/2.$

---

3. Show that :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(m\,x)\cos(n\,x) \;=\; 0$$

for all integer values of the constants m and n.

Solution : Starting with the sin addition formulae, we get :

$\sin(p\pm q) \;=\; \sin p\cos q \;\pm\; \sin q\cos p$

So that we can write the integrand as :

$$\sin(m\,x)\cos(n\,x) \;=\; \frac{1}{2}\left(\sin(m\;+n)\,x\;+\sin(m-n)\,x\right)$$

And our initial integral becomes :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(m\,x)\cos(n\,x) \;=\; \frac{1}{2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(m+n)\,x\;\;\sin(m-n)\,x\,dx \;=$$

$$\frac{-1}{4\pi}\left(\frac{\cos(m+n)\,x}{m+n}+\frac{\cos(m-n)\,x}{m-n}\right)\Bigg|_{-\pi}^{\pi}$$

Since cos is an even function, we know that $\cos\alpha\,\pi = \cos(-\alpha\,\pi)$, so these terms sum to zero when you evaluate them at the limits of integration. The only case that needs to be considered separately is when m = n = 0, since this gives rise to an indeterminate form (both the numerator and denominator tend toward zero when $m \to n$).

We can test this case by setting m = n in the original integrand:

$$\sin(m\,x)\cos(n\,x) \;=\; \sin(m\,x)\cos(m\,x) \;=\; \frac{1}{2}\sin(2\,m\,x)$$

Integrating this function between - π and π returns :

$$-\frac{1}{4}\cos(2\,m\,x)\Big|_{-\pi}^{\pi} = 0$$

from the even nature of the cos function. You could also apply L' Hopital' s rule to the indeterminate limit to obtain the same result.

---

4. Evaluate :

$$\int x^2 \cos(m\,x)\,dx$$

Solution : This (and the next integral) can be solved using integration by parts twice; you will see how the next integral is slightly nuanced.

You learned in Calc I that :

$$\int_\alpha^\beta u\,dv = uv\Big|_\alpha^\beta - \int_\alpha^\beta v\,du$$

In this case, we initially set :

$$u = x^2 \Rightarrow du = 2\,x\,dx$$
$$dv = \cos(m\,x)\,dx \Rightarrow v = \sin(m\,x)/m$$

Therefore, our integral becomes :

$$\int x^2 \cos(m\,x)\,dx = \frac{x^2 \sin(m\,x)}{m} - \frac{2}{m}\int x \sin(m\,x)\,dx$$

The integral on the right can also be attacked using integration by parts; in this case, u = x and dv = sin (m x) dx, so we get :

$$\int x^2 \cos(m\,x)\,dx = \frac{x^2 \sin(m\,x)}{m} - \frac{2}{m}\left(\frac{-x\cos(m\,x)}{m} - \frac{(-2)}{m}\int \cos(m\,x)\,dx\right) =$$

$$\frac{x^2 \sin(m\,x)}{m} + \frac{2\,x}{m^2}\cos(m\,x) - \frac{2}{m^3}\sin(m\,x)$$

Verifying with Mathematica :

In[68]:=
```
Integrate[x^2 Cos[m x], x]
```

Out[68]=

$$\frac{2\,x\,\text{Cos}[m\,x]}{m^2} + \frac{\left(-2 + m^2 x^2\right)\text{Sin}[m\,x]}{m^3}$$

---

5. Evaluate :

$$\int_0^\infty e^{ax}\cos(b\,x)\,dx$$

where a and b are real constants.

Solution : I will first solve for the indefinite integral since that is the real work of the problem, and evaluate between limits at the end. Since the integrand is a product, we can make use of integration by parts. Setting :

$$u = e^{ax}; \; du = a\, e^{ax}\, dx$$
$$dv = \cos bx; \; v = \sin bx / b$$

These substitutions yield :

$$\int e^{ax} \cos bx\, dx = \frac{1}{b} \cdot e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx\, dx$$

So far the process seems identical to the one we followed in question 4; the integral on the right can be integrated by parts, giving us :

$$\int e^{ax} \cos bx\, dx = \frac{1}{b} \cdot e^{ax} \sin bx - \frac{a}{b}\left[\frac{-1}{b} e^{ax} \cos bx - \left(\frac{-a}{b}\right) \int e^{ax} \cos bx\, dx\right] \quad (1)$$

But now we see the difference between this integral and the one in #4; we recognize that we could continue this process indefinitely, since the result of the integration by parts will always yield another integral. However, if we look carefully at the final integral in eq. (1), we see that apart from the multiplicative factor (a/b), it is the same integral we wanted to evaluate in the first place. So, let' s rewrite eq. (1) as :

$$\int e^{ax} \cos bx\, dx = \frac{1}{b} \cdot e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx\, dx$$

Treating the integrals algebraically, we rewrite this equation :

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx\, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx$$

Dividing through :

$$\int e^{ax} \cos bx\, dx = \frac{\frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx}{1 + \frac{a^2}{b^2}} = \frac{e^{ax} (b \sin bx + a \cos bx)}{a^2 + b^2}$$

Verifying via Mathematica :

```
In[79]:= Integrate[Exp[a x] Cos[b x], x]
```

$$\text{Out[79]=} \quad \frac{e^{a x} (a \,\text{Cos}[b\, x] + b \,\text{Sin}[b\, x])}{a^2 + b^2}$$

Alternate Solution:: If you are familiar with complex functions, you can solve this integral without doing any integration by parts. First, recall that when written in complex form,

$$e^{i z} = \cos z + i \sin z$$

where is the imaginary number. Thus, our original integral can be recast as :

$$\int e^{ax}\cos bx\,dx = \text{Re}\int e^{ax}e^{ibx}\,dx = \text{Re}\int e^{x(a+ib)}\,dx$$

where Re means we take the real part of the exponential. Now, integrating exponentials is much easier than multiple integrations by parts, and we get :

$$\int e^{ax}\cos bx\,dx = \text{Re}\left[\frac{1}{a+ib}\,e^{x(a+ib)}\right] = \text{Re}\left[\frac{e^{ax}(\cos bx + i\sin bx)}{a+ib}\right]$$

Multiplying numerator and denominator by the complex conjugate of the denominator :

$$\int e^{ax}\cos bx\,dx = \text{Re}\left[e^{ax}\,\frac{(\cos bx + i\sin bx)(a-ib)}{(a+ib)(a-ib)}\right] =$$

$$\text{Re}\left[e^{ax}\,\frac{(\cos bx + i\sin bx)(a-ib)}{a^2+b^2}\right] = \frac{e^{ax}}{a^2+b^2}\,\text{Re}[a\cos bx + b\sin bx + i(a\sin bx - b\cos bx)]$$

(Remember that $i = \sqrt{-1}$ so that $i^2 = -1$.) Since we only want the real part of this expression, we reproduce our result:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

As an added bonus, if we take the imaginary part of the expression above, we obtain the result :

$$\int e^{ax}\sin bx\,dx = \text{Im}\int e^{x(a+ib)}\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Evaluating the definite integral: To evaluate the definite integral, note that if $a > 0$ the integral will not converge. For $a < 0$, the function goes to zero as $x \to \infty$, so we get :

$$\int_0^\infty e^{ax}\cos bx\,dx = e^{ax}\,\frac{a\cos bx + b\sin bx}{a^2+b^2}\bigg|_0^\infty = 0 - \frac{a}{a^2+b^2} = \frac{-a}{a^2+b^2},\ a<0$$

Verifying through Mathematica (see if you can figure out the coding used in the Assumptions) :

```
In[81]:= Integrate[Exp[a x] Cos[b x], {x, 0, ∞}, Assumptions → Re[a] < 0 && b ∈ Reals]
```

Out[81]= $-\frac{a}{a^2+b^2}$

---

6. If s is the distance of a particle from the origin, find the period, amplitude, frequency and velocity amplitude for the following function :

$$s = 4\sin(3t+\pi/4) + 4\sin(3t-\pi/4)$$

Solution : We want to write this expression for displacement in terms of a single sin function. We do this by making use of the identity employed in question #3, which yields :

$$s = 4\sin(3t+\pi/4) + 4\sin(3t-\pi/4) = 4(2\sin(3t)\cos(\pi/4))$$

Cos (π/4) is time independent, so we can write the displacement function as

$$s = 8 \cos(\pi/4) \sin(3 t)$$

You know from your studies of the wave equation that the amplitude is the coefficient of the sinusoidal term, so A = 8 cos (π/4) = 5.66.

Your book shows that the period of a sin wave can be expressed as sin (2 π t/P) where P is the period; thus, the period of sin 3 t satisfies :

$$3 t = 2 \pi t/P \Rightarrow P = 2 \pi/3.$$

The frequency is 1/P, so the frequency = 3/2 π.
Velolcity is the time derivative of displacement, so we have :

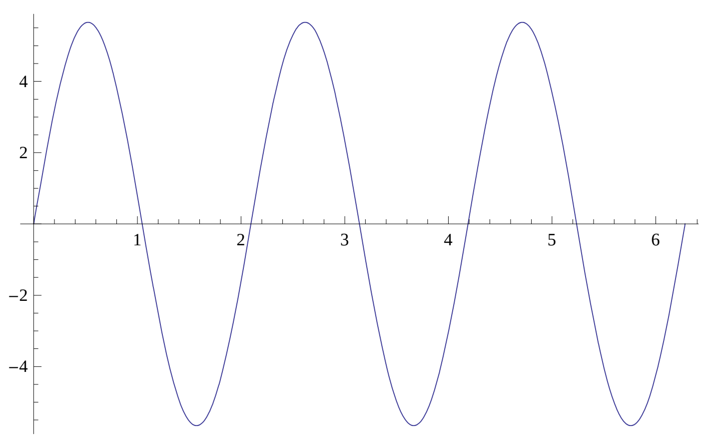
$$v = \frac{ds}{dt} = \frac{d}{dt}(8 \cos(\pi / 4) \sin 3 t) = 24 \cos(\pi / 4) \cos 3 t.$$

The amplitude of the velocity is the coefficient of the cos term, or 24 cos (π/4)

Let' s plot our original function to test our results :

In[85]:=
```
Plot[4 Sin[3 t + π / 4] + 4 Sin[3 t - π / 4], {t, 0, 2 π}]
```

Out[85]=



We can see that the amplitude is slightly greater than 5.5, consistent with our calculated result, and that our interval of 2 π contains exactly 3 wavelengths, showing the period is 2 π/3, confirming again our results.