PHYS 301
HOMEWORK #2--Solutions

On this homework assignment, you may evaluate integrals either by either direct integration, by employing symmetry arguments, or by citing previous results derived in this course. For instance, if one of the results from homework #1 helps evaluate an integral, you may cite that result and proceed. Display all work and/or provide your reasoning. You may use Mathematica to verify your results, but must submit complete work.

We will frequently use the basic equations pertaining to Fourier series for functions that are 2π periodic on $(-\pi, \pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n x + \sum_{n=1}^{\infty} b_n \sin n x$$

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n =$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx$$

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1. For the function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$

Find expressions for the Fourier coefficients and write the first three non zero terms of each expansion (use the format shown in the answer to problem 2 on page 354 of the text). Do all integrals by hand and show all work.

Solution : We use the standard equations to find the Fourier coefficients :

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos(n x)\,dx = \frac{1}{\pi}\left[\frac{x^2 \sin(n x)}{n} + \frac{2 x \cos(n x)}{n^2} - \frac{2 \sin(n x)}{n^3}\right]\Bigg|_0^{\pi}$$

Since we are evaluating at $x = 0$ and $x = \pi$, the sin terms vanish since $\sin(n\pi)$ is zero for all integer values of n; therefore :

$$a_n = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin(n\,x)\,dx = \frac{1}{\pi}\left[\frac{-x^2}{n}\cos(n\,x) + \frac{2\,x}{n^2}\sin(n\,x) + \frac{2}{n^3}\cos(n\,x)\right]\Bigg|_0^{\pi} =$$

$$\frac{1}{\pi}\left[\frac{-\pi^2}{n}(-1)^n + \frac{2}{n^3}((-1)^n - 1)\right]$$

We can summarize the b coefficients as :

$$b_n = \begin{cases} -\pi/n, & \text{n even} \\ \pi/n - 4\big/n^3\pi, & \text{n odd} \end{cases}$$

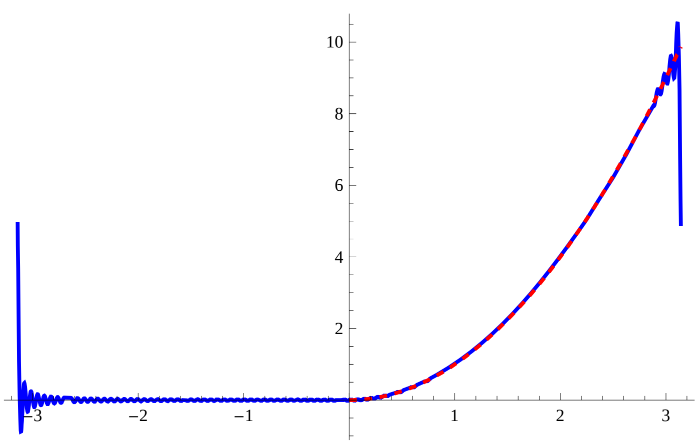
We can write the first three non zero terms of each series as :

$$f(x) =$$

$$\frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos 2\,x}{2^2} + \frac{\cos 3\,x}{3^2} - \ldots\right) + \left(\left(\pi - \frac{4}{\pi}\right)\sin x - \frac{\pi\,\mathrm{Sin}\ 2\,x}{2} + \left(\frac{\pi}{3} - \frac{4}{27\,\pi}\right)\sin 3\,x - \ldots\right)$$

Verifying with Mathematica: Using the analytical expressions for the coefficients, we can verify with Mathematica :

```
graph1 = Plot[π^2 / 6 + 2 Sum[(-1)^n Cos[n x] / n^2, {n, 1, 99}] +
     Sum[(-2 + (2 - n^2 π^2) (-1)^n ) Sin[n x] / (n^3 π), {n, 1, 99}],
    {x, -π, π}, PlotStyle → {Blue, Thick}];
graph2 = Plot[x^2, {x, 0, π}, PlotStyle → {Red, Dashed, Thick}];
Show[graph1, graph2]
```



In the graph above, I plot the curve of the original function (for x > 0) on the same set of axes as the graph derived from the Fourier expansion. To show that there really are two different curves, the graph of the Fourier expansion is represented by the continuous thick blue line, and the graph of y = x^2 is shown by the dashed red line.

2. For the function :

$$f(x) = \text{Abs}[x], \; -\pi < x < \pi$$

Find the Fourier coefficients and write out the first three non zero terms of the series expansion.

Solution :

First, a plot of this function will help our analysis greatly :

```
Plot[Abs[x], {x, -π, π}]
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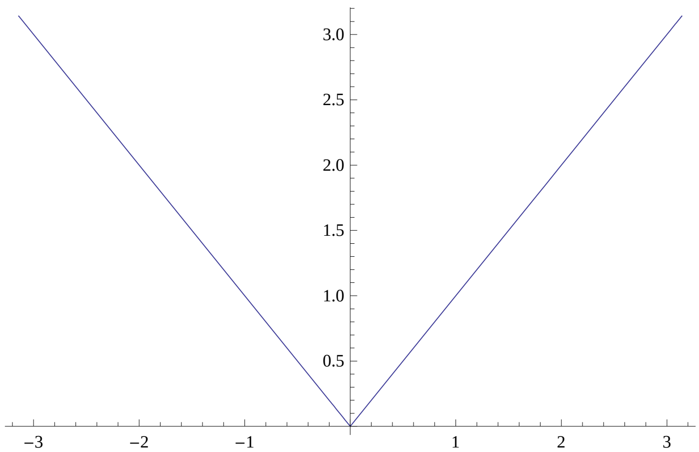

The graph shows that Abs[x] is an even function; making use of symmetry arguments, we can write immediately :

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(n x) \, dx = \frac{2}{\pi}\left[\frac{x \sin(n x)}{n} \bigg|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin(n x) \, dx \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2} \cos(n x) \bigg|_0^{\pi} \right] = \frac{2}{\pi n^2} \left((-1)^n - 1\right) = \begin{cases} 0, & \text{n even} \\ -4/\pi n^2, & \text{n odd} \end{cases}$$

$b_n = 0$; there can be no odd terms in the expansion series of an even function

Hence, the Fourier series for this function is :

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n x)}{n^2} = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \ldots \right]$$

Verifying through Mathematica :

```
Plot[(π / 2) - (4 / π) Sum[Cos[n x] / n^2, {n, 1, 31, 2}], {x, -3 π, 3 π}]
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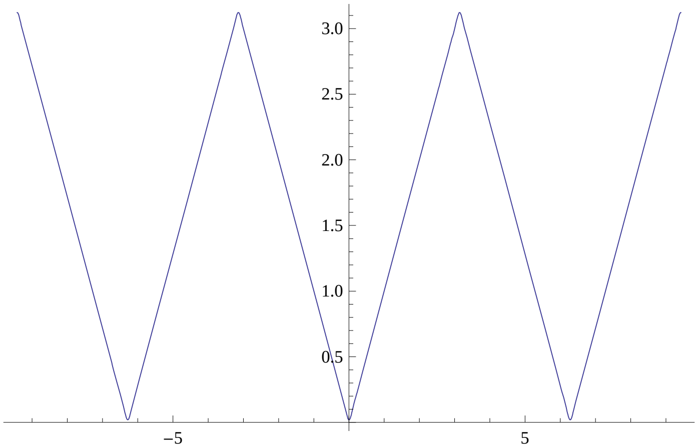


3. Find the Fourier coefficients and write out the first three non zero terms for :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin(2x), & 0 < x < \pi \end{cases}$$

Solution : We use the standard equations to find the Fourier coefficients :

$$a_o = \frac{1}{\pi}\int_0^\pi \sin(2x)\,dx = \frac{-1}{2\pi}(\cos(2n\pi) - 1) = 0$$

note that $\cos(2n\pi)$ is 1 for all integer values of n. For the a_n coefficients, use the trig identity from question 3 of HW 1 and a fair bit of algebraic manipulation to find: :

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(2x)\cos(nx)\,dx = \frac{1}{2\pi}\int_0^\pi (\sin(2+n)x + \sin(2-n)x)\,dx =$$

$$\frac{1}{2\pi}\left[\frac{-\cos(2+n)x}{2+n}\Big|_0^\pi + \frac{\cos(n-2)x}{n-2}\Big|_0^\pi\right] =$$

$$\frac{1}{2\pi}\left[\frac{(-1)^n - 1}{n-2} - \frac{(-1)^n - 1}{n+2}\right] = \frac{(-1)^n - 1}{2\pi}\left[\frac{1}{n-2} + \frac{1}{n+2}\right] = \frac{2[(-1)^n - 1]}{\pi(n^2-4)} = \begin{cases} 0, & \text{n even} \\ \frac{-4}{\pi(n^2-4)}, & \text{n odd} \end{cases}$$

Now, to find the b_n coefficients, we use the trig identity from problem 1 of HW 1 and we set :

$$b_n = \frac{1}{\pi}\int_0^\pi \sin(2x)\sin(nx)\,dx = \frac{1}{2\pi}\int_0^\pi \cos(n-2)x - \cos(n+2)x\,dx =$$

$$\frac{1}{2\pi}\left[\frac{\sin(n-2)x}{n-2} - \frac{\sin(n+2)x}{n+2}\right]\Big|_0^\pi$$

It is easy to see that setting $x = \pi$ or $x = 0$ yields zero in all cases EXCEPT for when n = 2. In this case, we calculate the single coefficient :

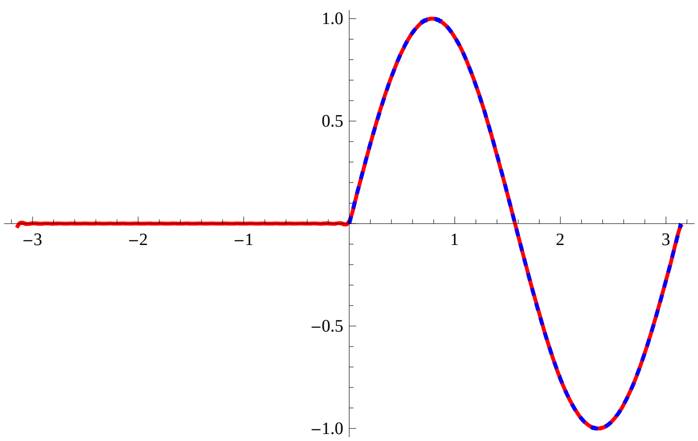
$$b_2 = \frac{1}{\pi}\int_0^\pi \sin(2x)\sin(2x)\,dx = \frac{1}{2}$$

Combining all these results, we get a Fourier series of :

$$f(x) = \frac{-4}{\pi} \sum_{n\,odd}^{\infty} \frac{\cos(n\,x)}{n^2-4} + \frac{\sin 2\,x}{2} = \frac{-4}{\pi}\left(\frac{\cos x}{1^2-4} + \frac{\cos 3\,x}{3^2-4} + \frac{\cos 5\,x}{5^2-4} + \ldots\right) + \frac{\sin 2\,x}{2}$$

Verifying: We verify this series by plotting the Fourier expansion and the curve y = sin 2 x on the same set of axes :

```
g1 = Plot[Sin[2 x] / 2 - (4 / π) Sum[Cos[n x] / (n^2 - 4), {n, 1, 51, 2}],
   {x, -π, π}, PlotStyle → {Red, Thick}];
g2 = Plot[Sin[2 x], {x, 0, π}, PlotStyle → {Blue, Dashed, Thick}];
Show[g1, g2]
```



And you see the two curves match for $0 < x < \pi$, and the Fourier expansion is zero for $-\pi < x < 0$.

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4. For $f(x) = \cos\alpha\,x$, $-\pi < x < \pi$ where α is not an integer, find the Fourier coefficients and first three non - zero terms of the expansion.

Solution : Using symmetry arguments with $\cos\alpha\,x$, an even function, we have :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha\,x = \frac{2}{\pi}\int_0^{\pi} \cos\alpha\,x\,dx = \frac{2}{\alpha\,\pi}\sin\alpha\,x\Big|_0^{\pi} =$$

$\frac{2}{\alpha\,\pi}\sin\alpha\,\pi$ (remember, α is not an integer so $\sin\alpha\,\pi$ is not zero)

Using the cos addition formula to write cos x cos y = 1/2 (cos (x + y) + cos (x - y)) :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha\,x\cos n\,x\,dx =$$

$$\frac{2}{\pi}\int_0^{\pi} \cos\alpha\,x\cos n\,x\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\cos(\alpha+n)\,x + \cos(\alpha-n)\,x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(\alpha+n)\,x}{\alpha+n} + \frac{\sin(\alpha-n)\,x}{\alpha-n}\right]\Bigg|_0^{\pi}$$

Expanding the sin expressions, realizing that the sin n π terms will be zero, and rearranging algebraically, we get :

$$a_n = \frac{1}{\pi} \frac{2\alpha}{\alpha^2 - n^2} \sin\alpha x \cos n x \Big|_0^\pi = \frac{2\alpha \sin\alpha\pi\,(-1)^n}{\pi\left(\alpha^2 - n^2\right)}$$

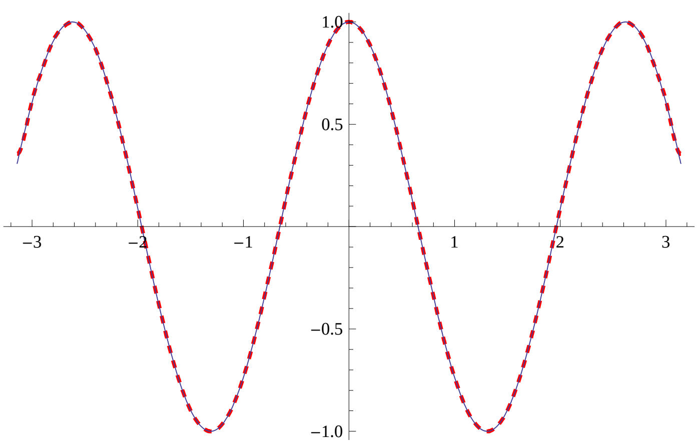
Employing symmetry again, we recognize that all the b_n coefficients must be zero since cos α x is an even function. Therefore, our Fourier series is :

$$f(x) = \cos\alpha x = \frac{\sin\alpha\pi}{\alpha\pi} + \frac{2\alpha\sin\alpha\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n x)}{\alpha^2 - n^2} =$$

$$\frac{\sin\alpha\pi}{\alpha\pi} - \frac{2\alpha\sin\alpha\pi}{\pi}\left(\frac{\cos x}{\alpha^2-1} + \frac{\cos 2x}{\alpha^2-4} + \frac{\cos 3x}{\alpha^2-9} + \ldots\right) \tag{1}$$

Verifying with Mathematica; I have superimposed the two graphs with the Fourier Series represented by thick red dashes so you can tell there are actually two curves. (Since Mathematica must have actual values to plot, I arbitrarily set α = 2.4, there is nothing special about this number, I just needed to give the program specfic input) :

```
α = 2.4;
graph1 = Plot[Sin[α π] / (α π) + (2 α Sin[α π] / π) Sum[(-1)^n Cos[n x] / (α^2 - n^2), {n, 1, 31}],
   {x, -π, π}, PlotStyle → {Red, Dashed, Thick}];
graph2 = Plot[Cos[α x], {x, -π, π}];
Show[graph1, graph2]
```



5. Use the results of question 4 to show that :

$$\pi\cot\alpha\pi - \frac{1}{\alpha} = 2\alpha \sum_{n=1}^{\infty} \frac{1}{\alpha^2 - n^2}$$

Solution : We start by setting x = π in eq. (1) above :

$$\cos\alpha\pi = \frac{\sin\alpha\pi}{\alpha\pi} + \frac{2\,\alpha\sin\alpha\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)}{\alpha^2-n^2}$$

But $\cos(n\pi) = (-1)^n$, so when $x = \pi$, the numerator inside the summation is simply 1, and we get :

$$\cos\alpha\pi = \frac{\sin\alpha\pi}{\alpha\pi} + \frac{2\,\alpha\sin\alpha\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{\alpha^2-n^2}$$

From this point, simple algebra yields our result, the partial fraction decomposition of the cotangent function :

$$\pi\cot\alpha\pi - \frac{1}{\alpha} = 2\,\alpha\sum_{n=1}^{\infty}\frac{1}{\alpha^2-n^2}$$

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6. Later in the course, we will study a series of orthogonal polynomials on (-1, 1) called Legendre polynomials. The first, second and third order Legendre polynomials are respectively :

$$P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2-1\right) \qquad P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$$

Show that these 3 Legendre polynomials satisfy orthogonality, namely :

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \neq 0, & m = n \end{cases}$$

If we wish to normalize the Legendre polynomials, i.e., :

$$c\int_{-1}^{1} P_m(x)\,P_m(x)\,dx = 1,$$

deduce the expression for the factor c which will satisfy orthonormality.

Solution : We can verify the orthogonality of the Legendre polynomials by direct integration :

$$\int_{-1}^{1} P_1(x)\,P_2(x)\,dx = \int_{-1}^{1} x\cdot\frac{1}{2}\left(3x^2-1\right)dx = \frac{1}{2}\int_{-1}^{1} 3x^3 - x\,dx = \frac{1}{2}\left(\frac{3}{4}x^4 - \frac{x^2}{2}\right)\Bigg|_{-1}^{1} = 0$$

Alternately, we could have used symmetry arguments. The integrand is an odd function (since the product of an odd function and an even function is odd), and the integral of an odd function between - L and L is zero. Similarly, we can use symmetry arguments to show that :

$$\int_{-1}^{1} P_2(x)\,P_3(x)\,dx = 0$$

Finally, by direct integration, we show that :

$$\int_{-1}^{1} P_1(x)\, P_3(x)\, dx = \int_{-1}^{1} x \cdot \frac{1}{2}\left(5x^3 - 3x\right) dx = \frac{1}{2}\int_{-1}^{1} 5x^4 - 3x^2\, dx = \frac{1}{4}\left(x^5 - x^3\right)\Big|_{-1}^{1} = 0$$

Now let' s consider integrals of the form :

$$c_m \int_{-1}^{1} P_m(x)\, P_m(x)\, dx = 1$$

For m = 1 :

$$c_1 \int_{-1}^{1} x \cdot x\, dx = 1 \Rightarrow c_1 \cdot \frac{2}{3} = 1 \Rightarrow c_1 = \frac{3}{2}$$

For m = 2 :

$$c_2 \int_{-1}^{1} \left[\frac{1}{2} \cdot \left(3x^2 - 1\right)\right]^2 dx = \frac{c_2}{4} \int_{-1}^{1} 9x^4 - 6x + 1\, dx = c_2\left(\frac{2}{5}\right) = 1 \Rightarrow c_2 = \frac{5}{2}$$

For m = 3 :

$$c_3 \int_{-1}^{1} \left[\frac{1}{2} \cdot \left(5x^3 - 3x\right)\right]^2 dx = \frac{c_3}{4} \int_{-1}^{1} 25x^6 - 30x^4 + 9x^2\, dx = c_3\left(\frac{2}{7}\right) \Rightarrow c_3 = \frac{7}{2}$$

From this pattern, we can deduce that we can produce orthonormal functions, i.e., functions satisfying :

$$c_m \int_{-1}^{1} P_m(x)^2\, dx = 1$$

if we set

$$c_m = \frac{2m + 1}{2}$$

Let' s see if our expression for c will yield the correct result for the tenth Legendre polynomial :

```
m = 10;
((2 m + 1) / 2) Integrate[LegendreP[10, x]^2, {x, -1, 1}]
```

```
1
```

And we obtain the expected result.